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ORDER STATISTICS FOR CORRELATED NON-IDENTICALLY DISTRIBUTED RANDOM VARIABLES

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Abstract

This paper deals with a problem in which the joint statistics of a set of N random variables are known. Based on this knowledge, we derive the joint probability density function (PDF) of the L largest random variables ($L \leq N$). The N random variables will be assumed to be correlated and non-identically distributed. Problems typical to this one are normally encountered in the performance analysis of a certain class of receivers used in multipath fading channels with correlated and unbalanced diversity branches. In this application, the receiver has access to N signal-to-noise ratio (SNR) random variables, and it has to make a symbol decision based on the largest L SNRs. This class of receivers is widely known as generalized selection combining (GSC) receivers.

Key Words

Joint density functions, joint distribution functions, Nakagami fading, generalized selection combining, diversity communications receivers

1. Introduction

A wireless communication channel is usually characterized by a randomly time-varying impulse response. This is a direct consequence of the continuously changing characteristics of the transmission medium. A mobile communications channel is a good example of this kind of channels. The signal transmitted through a mobile or wireless channel usually undergoes a series of reflections before reaching its final destination at the receiver. Therefore, it is normal that the receiver can receive multiple copies of the same signal through a multitude of paths (this is the multipath effect), all originating at the transmitter. In mathematical terms the received signal is usually modelled as having a randomly time-varying amplitude. This effect is known as fading. As explained briefly above, fading occurs in channels with time-varying characteristics, an example of which is represented by multipath channels.

Multipath fading can severely degrade error rate performance of wireless communication systems. A widely used statistical model of fading channels is the so-called Nakagami-m, introduced by Nakagami in [1], and derived in detail and proved to be appropriate as a model of fading channels in [2]. In this model the envelope of the received signal is modelled by a Nakagami-m random variable. This model has attracted a great deal of research interest recently [3–6]. In [3] the authors derive the symbol error probability for coherent detection of several types of M-ary modulation schemes over Nakagami fading channels using the maximal ratio combining (MRC) method. In [4] the authors use an approach based on the moment generating function (MGF) of the combined signal-to-noise ratio (SNR) to study generalized selection combining (GSC) schemes in independent Nakagami fading channels. In [5] the authors derive bit error probabilities of equal gain combining (EGC) with quadrature differential phase shift keying (QDPSK) modulation over correlated Nakagami fading channels. In [6] the author presents a generic characterization of Nakagami fading channels based on a three-parameter model. The parameters used in the channel characterization model proposed in [6] are the correlations between fading branches, average branch powers and a fading factor that quantifies the severity of fading. A lot more on Nakagami fading can be found in the open literature ([7–17] are some of the more important examples).

In other well-known fading channel models, the received signal envelope can be modelled by a Rayleigh or a Rician distributed random variable. The latter is usually used in channels where a strong line of sight component (direct path between the transmitter and the receiver) is present. An attractive feature of the Nakagami fading model is that both the Rayleigh and the Rician models can be treated as special cases of the more general case of Nakagami fading.

Whatever model is used to describe the signal-fading behaviour, the result is a random variation of the received signal envelope and phase. As a result, the received signal has a time-varying envelope. In any communication receiver, it is vital to have a maximum received signal

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power so that the error rate can be kept within acceptable ranges. Error rates naturally tend to be higher when the signal envelope becomes lower. This explains how fading is a source of receiver performance degradation.

Of the many approaches to countermeasure the effects of signal fading is the use of diversity [2, 18, 19]. The concept of diversity reception is as follows [20]: The receiver is supplied with several replicas of the same information signal transmitted over several fading channels. Therefore, the probability that all signal replicas will have simultaneous deep fading is considerably reduced. To illustrate this idea, let the received signal power over one of the diversity channels have a probability p of being small enough to cause a symbol error. Let's assume that there are L independent channels over which the signals are received. Hence, simple probability theory suggests that there is a probability p^L that the power received over each of the L channels will be small enough to cause a symbol error.

There is a variety of diversity schemes that can be chosen according to different system architectures and requirements [7–17]. Antenna diversity is known to be the simplest among diversity techniques [21, 22], and it could be used with virtually all systems. In an antenna diversity system the receiver employs multiple antennas to receive the transmitted signal. The antennas should be appropriately spaced to insure independent fading among the received signals. Alternatively, in a frequency diversity system [20] the same information message is transmitted over multiple carriers with properly chosen frequency separation. Carrier frequency separation should be adjusted to insure independent fading among the received signals. Other diversity methods include time diversity and the use of spread spectrum techniques [20].

A suitable combining technique has to be used in the receiver to take advantage of diversity reception and to attain a satisfactory level of performance [20, 23]. The most common measure of performance is the receiver symbol error rate. There are practical cases where an error rate of 10^{-3} is more than adequate (e.g., speech communications), while in other applications the same error rate is far too poor (e.g., data communications). As mentioned above, the received signal power is different over the different diversity channels (branches). As a result, a diversity receiver is presented with a set of different SNRs over the different diversity branches. The combining method in the receiver defines how the receiver uses the diversity channel outputs to make symbol decisions. Combining methods vary in both performance and complexity, where usually better performance is associated with more complexity.

GSC [24] is a combining method that presents a trade-off between combiner complexity and error rate performance. MRC provides optimum error rate performance at the cost of an appreciable amount of receiver complexity [20]. With MRC, knowledge of the channel-fading characteristics is a requirement. Signal fading can be mathematically represented as a random amplitude effect accompanied by a random phase shift. An MRC combiner assumes that these random amplitude and phase effects are perfectly known (or, alternatively, are accurately estimated). On the other hand, selection combining (SC) represents the simplest among combining schemes. Using SC, only the one branch with the largest SNR is used to form the decision variable upon which the final symbol decision is based. Obviously, this involves a substantial reduction in the receiver complexity while allowing a certain degree of performance loss. To reduce the performance degradation in using SC, GSC which selects the branches with the L largest instantaneous SNRs (from among a total of $N > L$ branches) can be used [25, 26]. The outputs of the selected branches can be combined using MRC or EGC. In the remaining part of this paper the L largest SNR random variables will be referred to as the L maxima.

As an example of the use of GSC, consider a system with $N = 4$ diversity branches. Let's assume that the $L = 2$ branch outputs with largest SNRs are to be combined. If the branch SNRs over a certain symbol period are {3.0, 1.1, 4.5, 2.6} dB, then the receiver combines the outputs of the first and third branches to form the symbol decision. In this case the combiner will ignore the outputs coming from the second and fourth branches.

In GSC the branch SNRs are gamma distributed because the envelopes are assumed to be Nakagami-m distributed. This can be easily verified through a random variable transformation taking into account that the SNR is proportional to the square of the envelope. With GSC, predetection [27] or postdetection [28] combining may be employed. If predetection combining is used, the signals are first combined and then detected. If postdetection combining is used, the signals are detected before combining. Predetection systems are usually more complex and costly than postdetection systems, while on the other hand, they involve less switching transients. Predetection combining is usually performed at the IF level, hence, sophisticated circuitry is generally required. This is due to the need to satisfy the requirement of coherent summation for MRC and EGC. The reference signal for phase comparison can be either one of the incoming signals, or the combined signal itself. Postdetection receivers are usually much simpler to implement; because no combining at the IF level is required.

Branch SNRs (e.g., rake fingers in the case of frequency diversity) are generally correlated. In the case of antenna diversity this is caused by spacing constraints of the antennas. Signals $r_k(t)$ and $r_l(t)$ received by two different antennas are said to be uncorrelated if they satisfy the relation:

$$\mathrm{E}[r_k(t)r_l^*(t)] = \mathrm{E}[r_k(t)]\mathrm{E}[r_l^*(t)] \tag{1}$$

This is the result of, but does not constitute a sufficient condition for, the two signals being statistically independent. Antenna spacing can theoretically be adjusted so that the signals have no mutual dependence, and hence, satisfy (1). However, practical implementation usually falls short of achieving this condition.

It should be emphasized here that the receiver bases its symbol decisions on the set of received (random) signals over its antennas. Branch correlation means that signal quantities received by different antennas cannot be statistically independent. The basic difference this makes in the analysis is that for correlated random quantities the joint density and distribution functions cannot be expressed as products of marginal density and distribution functions. Additionally, the SNRs are generally different (branches are unbalanced). This is due to the specific antenna geometry used and the difference in signal arrival times resulting from different path lengths.

The development of a direct relation between the statistics of the output of a GSC system and the statistics of its input offers an indispensable tool for the error rate performance analysis of wireless systems. The problem of obtaining the joint probability density function (PDF) of the L maxima based on the N-variate joint PDF of the combiner inputs is a non-classical order statistics problem in which the input random variables are generally correlated and non-identically distributed. Previous related research dealt mostly with uncorrelated input random variables [29].

2. Derivation of the L Maxima Joint Statistics

The statistics of the output L maxima will be derived based on the cumulative density function (CDF) of the input random variables, i.e., the SNR random variables available at the outputs of all diversity branches. Let's denote the N correlated non-identically distributed input SNR random variables by the vector $\overline{X} = [X_1, X_2, \ldots, X_N]$ with a joint PDF $f_{\overline{X}}(x_1, x_2, \ldots, x_N)$ and a joint CDF $F_{\overline{X}}(x_1, x_2, \ldots, x_N)$. The CDF of the first maximum Y_1 is given by:

$$\begin{aligned} F_{Y_1}(y_1) &= \Pr\{Y_1 \leq y_1\} \\ &= \Pr\{\max\nolimits_1(X_1, X_2, \ldots, X_N) \leq y_1\} \\ &= \Pr\{(X_1 \leq y_1) \cap (X_2 \leq y_1) \cap \cdots \cap (X_N \leq y_1)\} \end{aligned} \tag{2}$$

where the notation $\max_i(\cdot)$ is being used to denote the ith maximum. This simply means that there are $i-1$ random variables that are larger than $\max_i(\cdot)$. More compactly, (2) can be rewritten in the form:

$$F_{Y_1}(y_1) = F_{\overline{X}}(y_1, y_1, \ldots, y_1) \tag{3}$$

Note that there are N y_1's in the argument of $F_{\overline{X}}(\cdot)$ in (3). As an illustration, let's assume $N = 4$. In this case the CDF in (3) is equal to the probability that the largest random variable, which we call Y_1, is smaller than a value y_1. This is simply equal to the probability that all four random variables are smaller than y_1. Mathematically:

$$F_{Y_1}(y_1) = \Pr\{X_1 < y_1, X_2 < y_1, X_3 < y_1, X_4 < y_1\} \tag{4}$$

The joint CDF of the first two maxima is:

$$\begin{aligned} &F_{Y_1Y_2}(y_1, y_2) \\ &= \Pr\{(Y_1 \leq y_1) \cap (Y_2 \leq y_2)\} \\ &= \Pr\left\{\begin{array}{l} \max\nolimits_1(X_1, X_2, \ldots, X_N) \leq y_1 \\ \cap \left(\bigcup\limits_{\substack{(i_1, i_2, \ldots, i_{N-1}) = 1 \\ i_1 \neq i_2 \neq \cdots \neq i_{N-1}}}^{N} \max\nolimits_2(X_{i_1}, x_{i_2}, \ldots, X_{i_{N-1}}) \leq y_2 \right) \end{array}\right\} \end{aligned} \tag{5}$$

Alternatively:

$$\begin{aligned} &F_{Y_1Y_2}(y_1, y_2) \\ &= \Pr\left\{\begin{array}{l} \bigcap\limits_{j_1=1}^{N} (X_{j_1} \leq y_1) \\ \cap \left(\bigcup\limits_{\substack{(i_1, i_2, \ldots, i_{N-1}) = 1 \\ i_1 \neq i_2 \neq \cdots \neq i_{N-1}}}^{N} \bigcap\limits_{j_2=1}^{N-1} (X_{i_{j_2}} \leq y_2) \right) \end{array}\right\} \end{aligned} \tag{6}$$

The joint CDF $F_{Y_1Y_2}(y_1, y_2)$ is equal to the probability that the largest random variable Y_1 is smaller than y_1, while the second largest random variable Y_2 is smaller than y_2. When $N = 4$, this is equal to the probability that all four random variable are smaller than y_1, while the three random variables that do not include the largest are all smaller than y_2. Given that the largest random variable can be any one of the four, we easily find that:

$$\begin{aligned} &F_{Y_1Y_2}(y_1, y_2) \\ &= \Pr\left\{\begin{array}{l} (X_1 < y_1 \cap X_2 < y_1 \cap X_3 < y_1 \cap X_4 < y_1) \\ \cap \left\{\begin{array}{l} (X_1 < y_2 \cap X_2 < y_2 \cap X_3 < y_2) \\ \cup (X_1 < y_2 \cap X_2 < y_2 \cap X_4 < y_2) \\ \cup (X_1 < y_2 \cap X_3 < y_2 \cap X_4 < y_2) \\ \cup (X_2 < y_2 \cap X_3 < y_2 \cap X_4 < y_2) \end{array}\right\} \end{array}\right\} \end{aligned} \tag{7}$$

The joint CDF of the first three maxima is:

$$\begin{aligned} &F_{Y_1Y_2Y_3}(y_1, y_2, y_3) \\ &= \Pr\{(Y_1 \leq y_1) \cap (Y_2 \leq y_2) \cap (Y_3 \leq y_3)\} \\ &= \Pr\left\{\begin{array}{l} \bigcap\limits_{j_1=1}^{N} (X_{j_1} \leq y_1) \\ \cap \left(\bigcup\limits_{\substack{(i_1, i_2, \ldots, i_{N-1}) = 1 \\ i_1 \neq i_2 \neq \cdots \neq i_{N-1}}}^{N} \bigcap\limits_{j_2=1}^{N-1} (X_{i_{j_2}} \leq y_2) \right) \\ \cap \left(\bigcup\limits_{\substack{(i_1, i_2, \ldots, i_{N-2}) = 1 \\ i_1 \neq i_2 \neq \cdots \neq i_{N-2}}}^{N} \bigcap\limits_{j_3=1}^{N-2} (X_{i_{j_3}} \leq y_3) \right) \end{array}\right\} \end{aligned} \tag{8}$$

Generalizing the above simple cases we get the joint CDF of the first L maxima in the form:

$$F_{Y_1 Y_2,\ldots,Y_L}(y_1, y_2, \ldots, y_L) = \Pr\left\{\bigcap_{j=1}^{L} (Y_j \le y_j)\right\} = \Pr\left\{\bigcap_{l=1}^{L}\left(\bigcup_{\substack{(i_1, i_2, \ldots, i_{N-l+1}) = 1 \\ i_1 \ne i_2 \ne \cdots \ne i_{N-l+1}}}^{N} \bigcap_{j_l=1}^{N-l+1}\left(X_{i_{j_l}} \le y_l\right)\right)\right\} \quad (9)$$

This joint CDF is defined over $L!$ regions in the L dimensional hyperspace corresponding to the different permutations of the L maxima. These permutations represent the different orderings of the values of the L maxima. To illustrate this consider the case when we need to evaluate $F_{Y_1Y_2}(\alpha, \beta)$ and $F_{Y_1Y_2}(\beta, \alpha)$. Let's assume that $\alpha > \beta$. Obviously, the second maximum cannot be larger than the first maximum. However, it cannot be concluded from this that $F_{Y_1Y_2}(\beta, \alpha)$ should be identically equal to zero. This is so because the first and second maxima could be equal to some two values γ and δ, respectively, where $\delta < \gamma < \beta$.

We are seeking the PDF of the L maxima, which can be determined by differentiating the CDF, i.e.:

$$f_{\overline{Y}}(y_1, y_2, \ldots, y_L) = \frac{\partial^L}{\partial y_1 \partial y_2 \cdots \partial y_L} F_{\overline{Y}}(y_1, y_2, \ldots, y_L) \quad (10)$$

As can be easily seen from (10), only those terms in the CDF expression that depend on all the random variables $y_1, y_2, \ldots, y_L$ are relevant to finding the PDF because these are the terms that are not zeroed by differentiation. The condition that the required terms must satisfy is $y_1 > y_2 > \cdots > y_L$. Removing the irrelevant terms from $F_{Y_1Y_2,\cdots,Y_L}(\cdot)$ in (9), and denoting the remaining function by $\tilde{F}_{Y_1Y_2,\cdots,Y_L}(\cdot)$ we obtain:

$$\tilde{F}_{Y_1 Y_2,\ldots,Y_L}(y_1, y_2, \ldots, y_L) = \Pr\left\{\bigcup_{\substack{(i_1, i_2, \ldots, i_{L-1}) = 1 \\ i_1 \ne i_2 \ne \cdots \ne i_{L-1}}}^{N} \left(\begin{array}{l}(X_{i_1} \le y_1) \cap (X_{i_2} \le y_2) \cap \cdots \\ \quad \cap\, (X_{i_L} \le y_L) \cap \cdots \cap (X_{i_N} \le y_L)\end{array}\right)\right\} \quad (11)$$

where $\{i_L, i_{L+1}, \ldots, i_N\}$ are all distinct and different from $\{i_1, i_2, \ldots, i_{L-1}\}$. When $N = 4$ and $L = 2$ as in (7) the result is:

$$\tilde{F}_{Y_1Y_2}(y_1, y_2) = \Pr\left\{\begin{array}{l}\{(X_1 < y_1) \cap (X_2 < y_2) \cap (X_3 < y_2) \cap (X_4 < y_2)\} \\ \cup\, \{(X_2 < y_1) \cap (X_1 < y_2) \cap (X_3 < y_2) \cap (X_4 < y_2)\} \\ \cup\, \{(X_3 < y_1) \cap (X_1 < y_2) \cap (X_2 < y_2) \cap (X_4 < y_2)\} \\ \cup\, \{(X_4 < y_1) \cap (X_1 < y_2) \cap (X_2 < y_2) \cap (X_3 < y_2)\}\end{array}\right\} \quad (12)$$

Note that in (11), all random variables X_{i_l} for $l \ge L$ are compared to y_L, not to y_l. Using the union formula for non-mutually exclusive random variables yields:

$$\Pr\left\{\bigcup_{i=1}^{N}(X_i)\right\} = \sum_{i=1}^{N} \Pr\{X_i\} - \sum_{i=1}^{N}\sum_{\substack{j=1 \\ j \ne i}}^{N} \Pr\{X_i \cap X_j\} + \sum_{i=1}^{N}\sum_{\substack{j=1 \\ j \ne i}}^{N}\sum_{\substack{k=1 \\ k \ne i,j}}^{N} \Pr\{X_i \cap X_j \cap X_k\} - \cdots \quad (13)$$

Noting that any number of intersections of the events in (11) produces a term that is not dependent on all the random variables, and following the same reasoning used in deriving (11) we get the new function:

$$\tilde{\tilde{F}}_{Y_1,\ldots,Y_L}(y_1, y_2, \ldots, y_L) = \sum_{\substack{(i_1, i_2, \ldots, i_{L-1}) = 1 \\ i_1 \ne i_2 \ne \cdots \ne i_{L-1}}}^{N} \Pr\left\{\left(\begin{array}{l}(X_{i_1} \le y_1) \cap (X_{i_2} \le y_2) \cap \cdots \\ \cap\, (X_{i_L} \le y_L) \cap \cdots \cap (X_{i_N} \le y_L)\end{array}\right)\right\} = \sum_{\substack{(i_1, i_2, \ldots, i_{L-1}) = 1 \\ i_1 \ne i_2 \ne \cdots \ne i_{L-1}}}^{N} F_{X_{i_1}X_{i_2},\ldots,X_{i_{L-1}}X_{i_L},\ldots,X_{i_N}}\left(\underbrace{y_1, y_2, \ldots, y_{L-1}}_{L-1}, \underbrace{y_L, \ldots, y_L}_{N-(L-1)}\right) \quad (14)$$

Returning to our main problem of using (10) to determine the joint PDF of the L maxima, we conclude the manipulations above by writing:

$$\begin{aligned} f_{\bar{Y}}(y_1, y_2, \ldots, y_L) &= \frac{\partial^L}{\partial y_1 \partial y_2 \cdots \partial y_L} \tilde{F}_{\bar{Y}}(y_1, y_2, \ldots, y_L) \\ &= \frac{\partial^L}{\partial y_1 \partial y_2 \cdots \partial y_L} \tilde{\tilde{F}}_{\bar{Y}}(y_1, y_2, \ldots, y_L) \end{aligned} \quad (15)$$

Therefore, the only region in which the PDF has a non-zero value is also the region $y_1 > y_2 > \cdots > y_L$. Performing the differentiation in (15), the PDF is found to be equal to:

$$
f_{\overline{Y}}(y_1, y_2, \ldots, y_L) = \begin{cases} \dfrac{\partial^L}{\partial y_1 \partial y_2 \cdots \partial y_L} \displaystyle\sum_{\substack{(i_1, i_2, \ldots, i_{L-1}) = 1 \\ i_1 \neq i_2 \neq \cdots \neq i_{L-1}}}^{N} F_{X_{i_1} X_{i_2}, \ldots, X_{i_{L-1}} X_{i_L}, \ldots, X_{i_N}} \left(\underbrace{y_1, y_2, \ldots, y_{L-1}}_{L-1}, \underbrace{y_L, \ldots, y_L}_{N-(L-1)} \right), & y_1 > y_2 > \cdots > y_L \\ 0, & \text{otherwise} \end{cases} \tag{16}
$$

where $\{i_L, i_{L+1}, \ldots, i_N\}$ are all distinct and different from $\{i_1, i_2, \ldots, i_{L-1}\}$. When $N = 4$ and $L = 2$ as in (7) and (12) the result is:

$$
f_{Y_1 Y_2}(y_1, y_2) = \begin{cases} \dfrac{\partial^2}{\partial y_1 \partial y_2} \begin{bmatrix} F_{\overline{X}}(y_1, y_2, y_2, y_2) \\ + F_{\overline{X}}(y_2, y_1, y_2, y_2) \\ + F_{\overline{X}}(y_2, y_2, y_1, y_2) \\ + F_{\overline{X}}(y_2, y_2, y_2, y_1) \end{bmatrix}, & y_1 > y_2 \\ 0, & \text{otherwise} \end{cases} \tag{17}
$$

As an application of the PDF derivation based on (16), let's use the joint CDF of exponentially correlated Nakagami-m random variables, given in [30]. To avoid unnecessarily long expressions, while illustrating the use of our result, let's assume $N = 3$ and $L = 2$. The joint CDF of $\overline{X} = [X_1, X_2, X_3]$ in this case is given by [30]:

$$
F_{X_1 X_2 X_3}(r_1, r_2, r_3) = \frac{(1-\rho^2)^m}{\Gamma(m)} \times \sum_{i_1, i_2 = 0}^{\infty} \frac{(1+\rho^2)^{-[i_1+i_2+m]} \rho^{2(i_1+i_2)}}{\prod_{j=1}^{2} i_j! \Gamma(i_j + m)} q(r_1, r_2, r_3) \tag{18}
$$

where ρ is a constant that specifies the exponential cross-correlation among the elements of $\overline{X}$, $\Gamma(\cdot)$ is the Gamma function, $\gamma(\cdot)$ is the incomplete Gamma function and:

$$
q(r_1, r_2, r_3) = \gamma\left(i_1 + m, \frac{r_1^2}{2(1-\rho^2)}\right) \gamma\left(i_1 + i_2 + m, \frac{r_2^2}{2}\left(\frac{1+\rho^2}{1-\rho^2}\right)\right) \times \gamma\left(i_2 + m, \frac{r_3^2}{2(1-\rho^2)}\right) \tag{19}
$$

Modifying (17) for the case where $N = 3$ and $L = 2$,

$$
f_{Y_1 Y_2}(y_1, y_2) = \begin{cases} \dfrac{\partial^2}{\partial y_1 \partial y_2} \begin{bmatrix} F_{\overline{X}}(y_1, y_2, y_2) \\ + F_{\overline{X}}(y_2, y_1, y_2) \\ + F_{\overline{X}}(y_2, y_2, y_1) \end{bmatrix}, & y_1 > y_2 \\ 0, & \text{otherwise} \end{cases} \tag{20}
$$

Applying (20) to (18), we find that for $y_1 > y_2$:

$$
f_{Y_1 Y_2}(y_1, y_2) = \frac{(1-\rho^2)^m}{\Gamma(m)} \sum_{i_1, i_2 = 0}^{\infty} \frac{(1+\rho^2)^{-[i_1+i_2+m]} \rho^{2(i_1+i_2)}}{\prod_{j=1}^{2} i_j! \Gamma(i_j + m)} \times \frac{\partial^2}{\partial y_1 \partial y_2} [q(y_1, y_2, y_2) + q(y_2, y_1, y_2) + q(y_2, y_2, y_1)] \tag{21}
$$

Defining:

$$
\begin{aligned}
b_1 &= i_1 + m \\
b_2 &= i_1 + i_2 + m \\
b_3 &= i_2 + m \\
c_1 &= \frac{1}{2(1-\rho^2)} \\
c_2 &= \frac{1+\rho^2}{2(1-\rho^2)}
\end{aligned} \tag{22}
$$

and performing the differentiations in (21), we obtain after some manipulation:

$$
f_{Y_1 Y_2}(y_1, y_2) = \frac{4(1-\rho^2)^m}{y_1 y_2 \Gamma(m)} \sum_{i_1, i_2 = 0}^{\infty} \frac{(1+\rho^2)^{-[i_1+i_2+m]} \rho^{2(i_1+i_2)}}{\prod_{j=1}^{2} i_j! \Gamma(i_j + m)} \times \begin{bmatrix} ((c_1 y_1^2)^{b_1} (c_2 y_2^2)^{b_2} e^{-(c_1 y_1^2 + c_2 y_2^2)} \\ + (c_2 y_1^2)^{b_2} (c_1 y_2^2)^{b_1} e^{-(c_2 y_1^2 + c_1 y_2^2)}) \gamma(b_3, c_1 y_2^2) \\ + ((c_2 y_1^2)^{b_2} (c_1 y_2^2)^{b_3} e^{-(c_2 y_1^2 + c_1 y_2^2)} \\ + (c_1 y_1^2)^{b_3} (c_2 y_2^2)^{b_2} e^{-(c_1 y_1^2 + c_2 y_2^2)}) \gamma(b_1, c_1 y_2^2) \\ + ((c_1 y_1^2)^{b_1} (c_1 y_2^2)^{b_3} \\ + (c_1 y_1^2)^{b_3} (c_1 y_2^2)^{b_1}) e^{-c_1 (y_1^2 + y_2^2)} \gamma(b_2, c_2 y_2^2) \end{bmatrix} \tag{23}
$$

For the special case of independent but non-identically distributed input random variables, (16) becomes:

$$f_{\overline{Y}}(y_1, y_2, \ldots, y_L) = \begin{cases} \displaystyle\sum_{\substack{(i_1, i_2, \ldots, i_{L-1}) = 1 \\ i_1 \neq i_2 \neq \cdots \neq i_{L-1}}}^{N} f_{X_{i_1}}(y_1) f_{X_{i_2}}(y_2) \cdots f_{X_{i_L}}(y_L) \prod_{j=L}^{N} F_{X_{i_j}}(y_L), & y_1 > y_2 > \cdots > y_L \\ 0, & \text{otherwise} \end{cases} \tag{24}$$

where $\{i_L, i_{L+1}, \ldots, i_N\}$ are all distinct and different from $\{i_1, i_2, \ldots, i_{L-1}\}$. This agrees with the expression derived in [29]. Specializing this for the $N = 3$ and $L = 2$ case:

$$f_{Y_1 Y_2}(y_1, y_2) = \begin{cases} f_{X_1}(y_1) f_{X_2}(y_2) f_{X_3}(y_2) F_{X_2}(y_2) F_{X_3}(y_2) \\ + f_{X_1}(y_2) f_{X_2}(y_1) f_{X_3}(y_2) F_{X_1}(y_2) F_{X_3}(y_2) & y_1 > y_2 \\ + f_{X_1}(y_2) f_{X_2}(y_2) f_{X_3}(y_1) F_{X_1}(y_2) F_{X_2}(y_2), \\ 0, & \text{otherwise} \end{cases} \tag{25}$$

To summarize, we can use (16) to determine the joint PDF of the output L maxima of N correlated unbalanced (non-identically distributed) input random variables based on the joint CDF of the input N random variables. A good application where (16) can be very useful is in the performance analysis of WCDMA systems deploying GSC in a generalized fading channel with arbitrary correlation and unbalance between the diversity branches.

3. Conclusion

We derived what is – to the best of our knowledge – a novel direct analytical relation between the statistics of the GSC scheme output and input SNR random variables. The novelty of this relation lies basically in the fact that we have allowed the involved decision random variables to be correlated. There were no restrictions on the nature of this correlation, however. This makes the results of this research more general, and facilitates their use in a wide variety of problems. The formula derived in this paper is very useful in the performance study of wireless communication systems deploying GSC as a tradeoff between implementation complexity and near optimal receiver performance. An example has been provided to demonstrate the use of the new formula.

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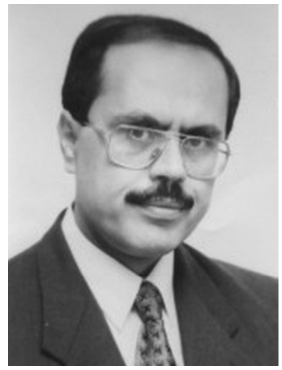

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