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A Novel Closed Form Moment Expression for Filtered Semiconductor Laser Phase Noise

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Summary

A novel closed form expression for the moments of a filtered semiconductor laser phase noise random variable is derived. These moments are essential for statistical characterization of filtered phase noise in heterodyne optical receivers, where some function of the phase noise process is usually integrated over a symbol interval. The derived expression enable s the computation of a given moment from the knowledge of the product of the laser linewidth and the symbol interval. The latter quantity will be referred to as the linewidth-duration product (LDP).

1 Introduction

Semiconductor laser phase noise plays a major role in degrading the performance of heterodyne optical communication receivers [1–4]. Most existing heterodyne receivers are based on matched filter models (or equivalently, integrate-and-dump filters) which are known to be optimal in the absence of laser phase noise [5]. Therefore, a substantial interest has grown over that past two decades or so in statistically characterizing filtered phase noise random variables [4], [6–14]. A brief review of these efforts is given in [15].

In a previous work [13] the author used simulated phase noise samples to obtain the log moment generating function (MGF) of a filtered phase noise random variable in the form of a finite-length power series of the frequency variable s. In a more recent work the author presented a much more accurate representation by expressing the power series in terms of $|s|^{1/4}$. As a matter of fact, the accuracy of such procedures is governed by the size of populations used in calculating the MGF. The number of the series coefficients, the choice of the independent variable (s, $|s|^{1/4}$, …, etc.) and the range over which this variable changes are also determining factors when judging the quality of the series representations.

This paper takes an analytical route to studying the statistics of filtered phase noise random variables. Several novel relations governing the random variables and their moments are derived. In particular, we derive a general formula for finding the nth order moment of a filtered phase noise random variable in terms of the LDP. Moments can be quite useful in MGF, characteristic function (CF), and probability density function (PDF) studies of filtered phase noise.

The results of this research are believed to represent a significant step towards a full analytical statistical representation of filtered phase noise. It is also believed that the results obtained here will be very helpful in performance studies of heterodyne optical receivers. In additions, they have the advantage of not suffering the limitations of small-phase noise approximations usually used in the literature.

2 Phase Noise Differential Equations

Let's define the linewidth duration factor (LDF) ρ of a phase noise process $\theta_\rho(t)$ by

$$\rho = \pi\beta T \tag{1}$$

where T is an integration interval (in a communication system, this is usually the duration of a bit or a symbol) and β is the linewidth of the phase noise process. Consider the random variable

$$\eta(\rho) = \frac{1}{T}\int_0^T e^{j\theta_\rho(t)}dt \tag{2}$$

It was shown in [4, 13, 14, 15] that the statistics of filtered phase noise random variables similar to the one in (2) are dependent on βT. Hence, such statistics must be also be dependent on ρ. The notation $\eta(\rho)$ is being used to signify this dependence. Let's consider the first derivative of $\eta(\rho)$ with respect to ρ, defined according to:

$$\frac{d\eta(\rho)}{d\rho} = \lim_{\varepsilon\to 0}\frac{\eta(\rho+\varepsilon)-\eta(\rho)}{\varepsilon} \tag{3}$$

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Since $\eta(\rho)$ is due to phase noise process $\theta_\rho(t)$ which has a linewidth β, then $\eta(\rho+\varepsilon)$ is due to a phase noise process with a linewidth $\beta+\delta$ such that (refer to (1) for comparison)

$$\rho+\varepsilon=\pi(\beta+\delta)T \tag{4}$$

Noting that $\theta_\rho(t)$ is zero-mean gaussian with variance $2\pi\beta t$ [4], it can be easily seen that the phase noise process needed to generate $\eta(\rho+\varepsilon)$ must be zero-mean gaussian with variance $2\pi(\beta+\delta)t$. In accordance with our terminology, the latter process can be denoted by $\theta_{\rho+\varepsilon}(t)$. The process $\theta_\rho(t)$ can be usually written in the form [4]

$$\theta_\rho(t)=2\pi\int_0^t \zeta_\rho(\tau)d\tau \tag{5}$$

where $\zeta_\rho(t)$ is a zero-mean white gaussian random process with double-sided power spectral density $\beta/2\pi$. Hence, the process $\theta_{\rho+\varepsilon}(t)=\theta_\rho(t(\beta+\delta)/\beta)$ (which is obviously zero-mean gaussian with variance $2\pi(\beta+\delta)t$ can be shown to be the process needed to generate $\eta(\rho+\varepsilon)$.

Therefore,

$$\eta(\rho+\varepsilon)=\frac{1}{T}\int_0^T e^{j\theta_\rho\left(\frac{\beta+\delta}{\beta}t\right)}dt \tag{6}$$

This can be manipulated to yield

$$\eta(\rho+\varepsilon)=\frac{\rho}{\rho+\varepsilon}\left[\eta(\rho)+\frac{1}{T}\int_0^{T\frac{\delta}{\beta}} e^{j\theta_\rho(t+T)}dt\right] \tag{7}$$

Noting that the interval $[0, T\,\delta/\beta]$ is supposed to vanish as ε tends to zero, we can easily assume that the exponent in the integrand in (7) is equal to the time-independent quantity $j\theta_\rho(T)$, which allows (7) to be approximated by:

$$\eta(\rho+\varepsilon)\approx\frac{\rho}{\rho+\varepsilon}\eta(\rho)+\frac{\varepsilon}{\rho+\varepsilon}e^{j\theta_\rho(t)} \tag{8}$$

Applying the limit in (3) produces

$$\frac{d\eta(\rho)}{d\rho}=\frac{1}{\rho}\left[e^{j\theta_\rho(T)}-\eta(\rho)\right] \tag{9}$$

This novel result is expected to have very important roles in a wide range of statistical analysis of filtered phase noise. Solving the first order differential equation in (9) yields

$$\eta(\rho)=\frac{1}{\rho}\int_0^\rho e^{j\theta_\xi(T)}d\xi \tag{10}$$

By a simple substitution of variables we can deduce (2) from (10), which verifies the derivation of (9). Furthermore,

$$\frac{d}{d\rho}\left(\eta^n\right)=n\eta^{n-1}(\rho)\frac{1}{\rho}\left[e^{j\theta_\rho(t)}-\eta(\rho)\right] \tag{11}$$

Let $\mu_n(\rho)$ denote the nth moment of $\eta(\rho)$ given by

$$\mu_n(\rho)E\left[\eta^n(\rho)\right] \tag{12}$$

Applying statistical expectation to both sides of (11) yields

$$\mu_n'(\rho)+\frac{n}{\rho}\mu_n(\rho)=\frac{n}{\rho}E\left[\eta^{n-1}(\rho)e^{j\theta_\rho(T)}\right] \tag{13}$$

where the prime in the superscript denotes differentiation with respect to ρ. The differential equation in (13) is particularly simple when $n=1$ in which case we have

$$\rho\mu_1'(\rho)+\mu_1(\rho)=e^{-\rho} \tag{14}$$

where we have used the fact that [4]

$$E\left[e^{j\theta_r(T)}\right]=e^{-\rho} \tag{15}$$

Solving (14) results in

$$\mu_1(\rho)=\frac{1}{\rho}\left(1-e^{-\rho}\right) \tag{16}$$

This is identical to the result obtained in [13] by directly applying statistical expectation to both sides of (2). The solution is not as easy when $n>1$, in which case $\eta^{n-1}(\rho)$ and $\theta_\rho(T)$ are not statistically independent. To perform the expectation on the right hand side of (13) we can first write

$$\begin{aligned}\eta^{n-1}(\rho)e^{j\theta_\rho(T)}&=\left(\eta(\rho)e^{j\frac{\theta_\rho(T)}{n-1}}\right)^{n-1}\\&=\left(\frac{1}{T}\int_0^T e^{jQ(t)}dt\right)^{n-1}\end{aligned} \tag{17}$$

where

$$Q(t)=\theta_\rho(t)+\frac{\theta_\rho(T)}{n-1} \tag{18}$$

Returning to (13) and using (17) we can express the mo ment differential equation for $n>1$ in the form

$$\mu_n'(\rho)+\frac{n}{\rho}\mu_n(\rho)=\frac{n}{\rho}\mu_{Q,n-1}(\rho) \tag{19}$$

where $\mu_{Q,n-1}(\rho)$ is the $(n-1)$st moment of $\eta_Q(t)$ given by:

$$\eta_Q(t)=\frac{1}{T}\int_0^T e^{jQ(t)}dt \tag{20}$$

Now, (19) can be solved to yield

$$\mu_n(\rho) = n\rho^{-n}\int_0^{\rho}\xi^{n-1}e^{-\xi}\mu_{Q,n-1}(\xi)d\xi \tag{21}$$

Making use of (5) we can write (18) in the form

$$Q(t) = \frac{1}{n-1}\left[2\pi(n-1)\int_0^t \zeta_\rho(\tau)d\tau + 2\pi\int_0^T \zeta_\rho(\tau)d\tau\right] \tag{22}$$

Splitting the rightmost integral into two integrals over the intervals [0, t] and [t, T] we obtain

$$Q(t) = \frac{1}{n-1}\left[n\theta_\rho(t) + \lambda(t)\right] \tag{23}$$

where

$$\lambda(t) = 2\pi\int_t^T \zeta_\rho(\tau)d\tau \tag{24}$$

Note that $\lambda(t)$ is a zero-mean gaussian random process which has a variance $2\pi\beta(T-t)$ and which is statistically independent of $\theta_\rho(t)$. Hence, Q(t) is zero-mean gaussian with variance

$$\sigma_Q^2 = \frac{(n^2-1)2\pi\beta t + 2\pi\beta T}{(n-1)^2} \tag{25}$$

The same variance in (25) is possessed by the zero-mean gaussian process

$$R(t) = \theta_\upsilon(t) + \frac{\varphi_\rho(T)}{n-1} \tag{26}$$

where

$$\upsilon = \frac{n+1}{n-1}\rho \tag{27}$$

and $\varphi_\rho(T)$ is a zero-mean gaussian random variable which has a variance $2\pi\beta T$, and which is statistically independent of $\theta_\upsilon(t)$ Therefore, the following random variable has the same statistics as η_Q

$$\eta_R = \frac{1}{T}\int_0^T e^{jR(t)}dt = e^{j\frac{\varphi_\rho(T)}{n-1}}\eta_\upsilon \tag{28}$$

where

$$\eta_\upsilon = \frac{1}{T}\int_0^T e^{j\theta_\upsilon(t)}dt \tag{29}$$

As a result,

$$\mu_{Q,n-1}(\rho) = e^{-\rho}\mu_{n-1}(\upsilon) = e^{-\rho}\mu_{n-1}\left(\frac{n+1}{n-1}\rho\right) \tag{30}$$

Substituting this relation in (21) yields

$$\mu_n(\rho) = n\rho^{-n}\int_0^{\rho}\xi^{n-1}e^{-\xi}\mu_{n-1}\left(\frac{n+1}{n-1}\right)d\xi \tag{31}$$

Successive substitutions of $\mu_{n-1}(\rho)$, $\mu_{n-2}(\rho)$, …,$\mu_1(\rho)$ into (31) obviously produce a general expression for $\mu_n(\rho)$. However, this leads to lengthy results which cannot be easily simplified.

Alternatively, we choose to start by substituting (30) into (19) to obtain

$$\mu_n'(\rho) + \frac{n}{\rho}\mu_n(\rho) = \frac{n}{\rho}e^{-\rho}\mu_{n-1}(a_n\rho) \tag{32}$$

where we have defined

$$a_n = \frac{n+1}{n-1} \tag{33}$$

Multiplying both sides of (32) by ρ^n gives after some rearranging

$$\frac{d}{d\rho}g_n(\rho) = \frac{n}{a_n^{n-1}}e^{-\rho}g_{n-1}(a_n\rho) \tag{34}$$

where

$$g_n(x) = x^n\mu_n(x) \tag{35}$$

Applying Laplace transform to (34) (by transforming ρ into s) yields

$$G_n(s) = \frac{n}{a_n^n s}G_{n-1}\left(\frac{s+1}{a_n}\right) \tag{36}$$

where $G_1(s)$ is the Laplace transform of $g_1(\rho)$. Now, from (16) and (35) we get

$$G_1(s) = \frac{1}{s(s+1)} \tag{37}$$

Using (36) to determine $G_2(s)$ from $G_1(s)$, then $G_3(s)$ from $G_2(s)$ and so on results in the following general expression for $G_n(s)$

$$G_n(s) = \frac{n!}{\prod_{k=0}^{n}(s+A_{n,k})} \tag{38}$$

where $A_{n,0} = 0$ and $A_{n,1} = 1$, while for $k \geq 2$ we have

$$A_{n,k} = A_{n,k-1} + \prod_{l=0}^{k-2}a_{n-1} \tag{39}$$

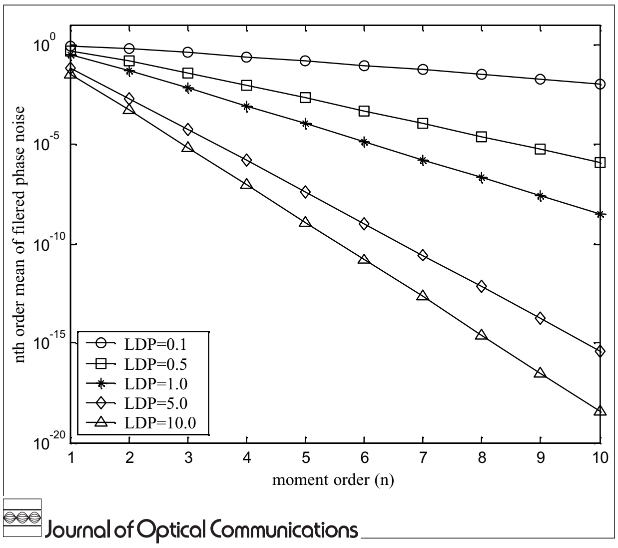

Fig. 1: nth order mean of filtered phase noise for several values of the LDP

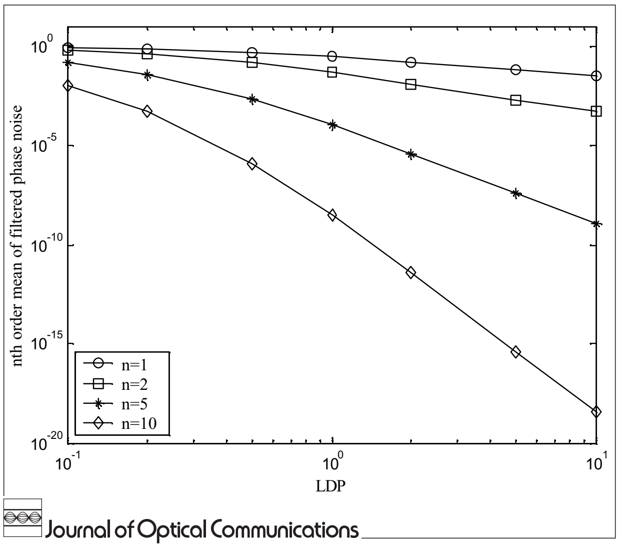

Fig. 2: nth order mean of filtered phase noise as function of the LDP for several values of n

Since the elements of the set $\{A_{n,k}\}$, which are the negatives of the poles of $G_n(s)$ in (38) are all distinct, as can be seen from (39), partial fractions expansion can be used to find that $g_n(\rho)$ is equal to

$$g_n(\rho) = n!\sum_{k=0}^{n} B_{n,k} e^{-A_{n,k}\rho} \tag{40}$$

where

$$B_{n,l} = \frac{1}{\prod\limits_{\substack{k=0 \\ k\neq l}}^{n} \left(A_{n,k} - A_{n,l}\right)} \tag{41}$$

Therefore, using (35) and (40)

$$\mu_n(\rho) = n!\rho^{-n} \sum_{k=0}^{n} B_{n,k} e^{-A_{n,k}\rho} \tag{42}$$

The following few moments were obtained by direct application of (42) and also by successive substitutions in (31):

$$\mu_2(\rho) = \frac{1}{6\rho^2}\left(3 - 4e^{-\rho} + e^{-4\rho}\right) \tag{43}$$

$$\mu_3(\rho) = \frac{1}{72\rho^3}\left(16 - 27e^{-\rho} + 12e^{-3\rho} - e^{-9\rho}\right) \tag{44}$$

$$\mu_4(\rho) = \tag{45}$$

$$\frac{3}{4000\rho^4\left(125 - 256e^{-\rho} + 162e^{-8\rho/3} - 32e^{-6\rho} + e^{-16\rho}\right)}$$

3 Results

The nth order mean of $\eta(\rho)$ has been plotted in Fig. 1 as function of n for several values of the LDP. A few observations can be made regarding this figure. First, since n is an integer, the curves should have been shown as sets of isolated points. However, we chose to connect the points to show the general trend in these curves as n is varied. The points themselves are indicated using markers. Second, the moments get smaller as the LDP gets larger. This observation is rather important; because it is well-known that η (or a related quantity) usually multiplies the signal term at the input of the symbol (or bit) decision circuitry in the receiver. Hence, a larger LDP results in smaller multipliers, leading to more frequent errors in the decision process. Third, and because of the second observation, if the moments are to be used in calculating the MGF or the CF of η, then fewer moments are needed as the LDP gets larger. Fourth, all curves in Fig. 1 are very close to linear curves. Noting that the vertical axis is logarithmic leads to a general conclusion that the dependence of the moments on n has a decaying exponential nature. This suggests that (42) is easy to approximate by a single decaying exponential term (in n) when a more involved analysis of the statistics of η is needed.

In Fig. 2 we plot the nth order mean of $\eta(\rho)$ as function of the LDP for several values of n. The second and third observations made about Fig. 1 can also be made about Fig. 2. However, it is obviously difficult to observe a pure decaying exponential dependence of the moments on the LDP. Therefore, even using an approximation, several exponential terms (rather than just one term) in (42) are necessary.

4 Conclusions

A novel filtered phase noise differential equation, with the LDF acting as an independent variable, has been derived. This equation can be very helpful in more thorough statistical modeling of filtered phase noise. A novel closed form expression for the moments of a filtered semiconductor laser phase noise random variable has also been derived. The dependence of the moments on both the moment order and the LDP has been illustrated. Decaying exponential approximations of moments as functions of moment orders appear to be reasonable. Such approximations can be extremely useful in MGF and CF (and consequently PDF) characterizations of filtered phase noise. This last reasoning in particular seems to need further investigation, though.

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