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MOMENT GENERATING FUNCTIONS OF FILTERED PHASE NOISE IN HETERODYNE OPTICAL RECEIVERS

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Abstract

In an earlier article [1] the author presented an accurate statistical characterization of filtered laser phase noise in heterodyne optical receivers. The author used simulated phase noise samples to obtain the natural logarithm (log) moment generating function (MGF) of filtered phase noise as a finite-length power series in the frequency variable s. Least squares curve fitting has been used to estimate the series coefficients. In this paper a much more accurate representation is obtained by expressing the power series in terms of $|s|^{1/4}$.

The results of this research are applicable to a host of problems in heterodyne optical receiver performance evaluation. They also have the advantage of being accurate in large- as well as small-phase noise situations. A wide range of phase noise levels is studied through the variation of the linewidth-duration product (values between 0.001 and 10 are considered). Most current and previous literature on this issue assumes phase noise is small. A demonstration of the error in applying the small phase noise approximation to cases of large and medium phase noise were presented in [1].

Key Words
Phase Noise, MGF, Heterodyne Detection.

1. Introduction

A major source of performance degradation in heterodyne optical receivers is semiconductor laser phase noise [2]-[5]. Most existing literature on system performance in the presence of phase noise assumes a matched-filter-based model for the demodulator, even though his model has not been proved to be optimal in terms of error probability. The lack of a phase-noise-optimal heterodyne optical receiver is due to the fact that phase noise is very difficult to characterize statistically. No closed form expressions are available for its probability density functions (PDFs) or moment generating functions.

The need for a good statistical model of filtered phase noise is due to the dominance of matched-filter-based demodulators in heterodyne optical receivers. The literature on attempts to statistically model filtered phase noise is too huge to cite here. However, a significant deal of this literature is devoted to small phase noise assumptions. Examples include the work by Einarsson [5] and by Foschini and Vannucci [6]. The weakness of small phase noise approximations becomes significant when they are applied to situations where phase noise cannot be assumed small. Substantial performance evaluation errors are inevitable when such approximations are used in these cases.

The random variable representing laser phase noise at the output of a matched filter (or an equivalent correlator) is most often given by:

$$\eta = \frac{1}{T}\int_0^T e^{j\theta(t)}dt \qquad (1)$$

where T is the integration interval (usually related to a symbol, bit or chip duration), and $\theta(t)$ is a Wiener-Levy laser phase noise random process with linewidth β. A common mathematical representation of $\theta(t)$ is

$$\theta(t) = 2\pi\int_0^t \varphi(\zeta)d\zeta \qquad (2)$$

where $\varphi(.)$ is a wide sense stationary gaussian random process having a zero mean and a flat power spectral density equal to $\beta/2\pi$ [5].

Equation (1) shows that η is a complex quantity with a real part

$$\eta_c = \frac{1}{T}\int_0^T \cos\left[\theta(t)\right]dt \qquad (3)$$

and an imaginary part

$$\eta_s = \frac{1}{T}\int_0^T \sin\left[\theta(t)\right] dt \qquad (4)$$

Hence, in complex notation

$$\eta = \eta_c + j\,\eta_s \qquad (5)$$

The magnitude squared of η is given by:

$$\begin{aligned} U &= \left|\eta\right|^2 \\ &= \eta_c^2 + \eta_s^2 \end{aligned} \qquad (6)$$

The random variables in (1), (3), (4), (5) and (6), in addition to $\left|\eta\right|$ are usually encountered in error probability analysis of heterodyne optical receivers that are based on matched filters or correlators. The most followed procedure in such analysis is to condition the error probability on the phase noise random variable, then to perform averaging using simulation. The author has presented an alternative to this process [1] by providing a semi-closed form expression for the MGF of U (a similar approach can be applied to the MGF or PDF of any of the other above-mentioned random variables). The random variable U in particular usually appears in conditional error probability expressions of FSK and DPSK receivers.

The objective of the present paper is to use the accurate phase noise simulation procedure outlined and used in [1] to generate phase noise samples. The paper then extends the work in [1] by representing the log MGF of U as a power series in $|s|^{1/4}$ rather than s. The motivation for using this new variable is two fold: first, the power series representation in [1] is not very accurate for large values of s, second, the new variable appeared in a very good analytical work on a small phase noise approximation given by Einarsson in [5]. A wide range of phase noise levels is studied through the variation of the linewidth-duration product βT (values between 0.001 and 10 are considered).

2. Mathematical Modeling of the MGF

The moment generating function of the random variable U is computed using direct numerical evaluation of the equation

$$\Psi_U(s) = E\left[e^{sU}\right] \qquad (7)$$

where the statistical average involved is commonly replaced by a population average. Throughout the results given in this paper, populations of 500,000 samples are used in every averaging operation. Because of the exponential dependence of $\Psi_U(s)$ on s, most of our work will be in terms of the log MGF, defined as

$$\Gamma_U(s) = \ln \Psi_U(s) \qquad (8)$$

It is well-known that the MGF of U can be written as an infinite power series in s, where the series coefficients are the moments of U. This representation has the form

$$\Psi_U(s) = \sum_{n=0}^{\infty} \frac{E\left[U^n\right] s^n}{n!} \qquad (9)$$

This representation has a great mathematical value, however, making use of it in numerical procedures usually requires an unnecessarily large number of moments to be pre-computed. Therefore, it has a limited value in situations where numerical procedures are necessary.

To get around the difficulty in using an MGF expression like the one in (9), an alternative representation is needed. This is done by obtaining a power series representation of $\Gamma_U(s)$ which involves a small number of coefficients.

Note that the proposed modeling of the phase noise MGF below is valid only for real values of the complex frequency variable s. This does not involve any loss of generality as far as the computation of error probabilities of many types of heterodyne optical receivers is concerned. The procedure can be easily generalized to take care of cases where s must remain complex.

We will assume that the log MGF can be written in the form

$$\Gamma_U(s) = \sum_{n=1}^{N} a_n z^n \qquad (10)$$

where

$$z = \operatorname{sgn}(s)\left|s\right|^{1/4} \qquad (11)$$

Note that the sign function had to be used in (11) because $\Psi(s)$ and $\Gamma_U(s)$ are not even functions of s, as will be demonstrated by plots of $\Gamma_U(s)$.

As will be seen from the results, (10) produces a sufficiently accurate representation of $\Gamma_U(s)$ and $\Psi_U(s)$ for values of N that are usually less than 20. Note that the sum in (10) does not include an $n = 0$ term to comply with the fact that $Y(0) = 1$. The emphasis on the MGF of $\boldsymbol{U}$ rather than its PDF is due to the direct use of the MGF in expressions of the error probability of optical heterodyne receivers utilizing noncoherent demodulators.

3. Results

As mentioned earlier, the results below are based on an accurate simulation of the magnitude squared filtered phase noise random variable. The simulation procedure was outlined in [1]. Each result is based on a population of 500,000 samples of $\boldsymbol{U}$.

Figure 1 shows the estimated PDF of $\boldsymbol{U}$ when the linewidth duration product varies from 0.1 to 10. Smaller values of b$\boldsymbol{T}$ are indicative of small phase noise, while larger values are indicative of severe phase noise. Note that, for example, a value of b$\boldsymbol{T} = 5$ means that the laser linewidth is five times the bit rate. Curves in Figure 1 demonstrate that small linewidth products result in values of $\boldsymbol{U}$ that are very close to unity. On the other hand, large linewidth duration products result in values of $\boldsymbol{U}$ that are very close to zero. These trends agree with the fact that filtered phase noise is a multiplicative type of noise, keeping in mind that the ideal phase noise case is equivalent to a phase noise with a zero linewidth duration product.

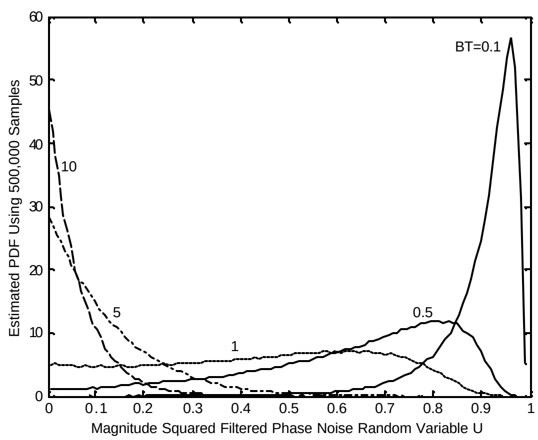

Figure 1: Estimated PDF of the magnitude-squared filtered phase noise random variable $\boldsymbol{U}$ for b$\boldsymbol{T}$ = 0.1,0.5,1,5,10

Figure 2 and Figure 3 show the estimated log MGF of $\boldsymbol{U}$ when b$\boldsymbol{T}$ varies from 0.001 to 10. Note that for very small phase noise the log MGF tends to approximate a linear function of $\boldsymbol{s}$. With little mathematical manipulation this leads to a PDF that approximates a delta function at $\boldsymbol{u} = 1$. Obviously, as b$\boldsymbol{T}$ gets smaller and smaller, the delta function approximation of the PDF becomes a better one with its center getting nearer and nearer to $\boldsymbol{u} = 1$, ultimately coinciding with the no phase case of $\boldsymbol{u} = 1$.

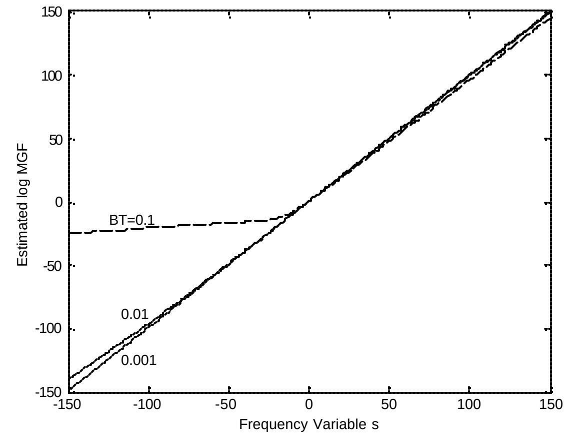

Figure 2: Estimated log MGF of the magnitude-squared filtered phase noise random variable $\boldsymbol{U}$ for b$\boldsymbol{T}$ = 0.001,0.01,0.1

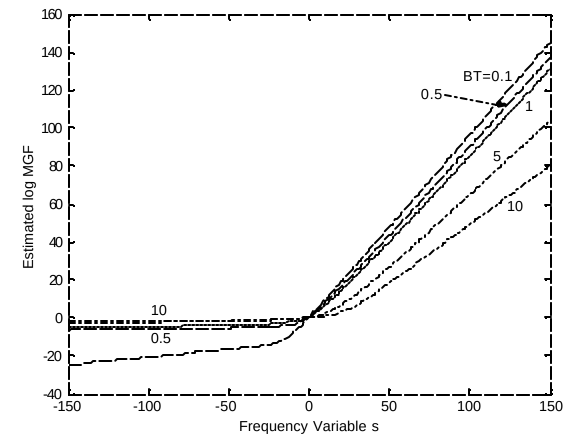

Figure 3: Estimated log MGF of the magnitude-squared filtered phase noise random variable $\boldsymbol{U}$ for b$\boldsymbol{T}$ = 0.1,0.5,1,5,10

Figure 4 shows the squared error in fitting ¡$(\boldsymbol{s})$ by the polynomial in (10), plotted versus $\boldsymbol{s}$. This error can be easily seen to be small enough to make the fitting process a suitable one.

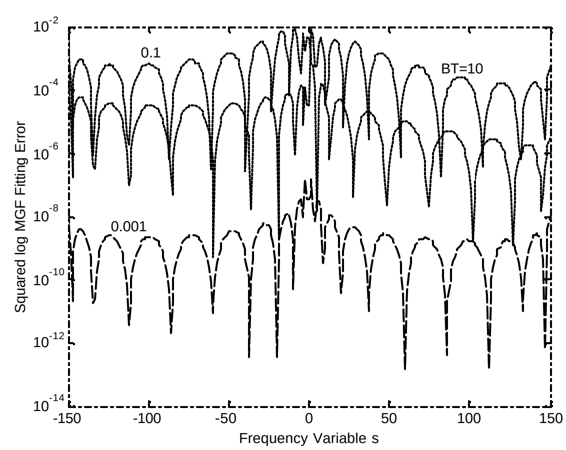

Figure 4: Least squares curve fitting error in the power series expansion of the log MGF of the magnitude-squared filtered phase noise random variable $\boldsymbol{U}$ for b$\boldsymbol{T}$ = 0.001,0.1,10

Finally, Table 1 gives all coefficient values in (10) for $N = 20$, and a wide range of the linewidth duration product. It should be emphasized that these values are the outcome of curve fitting based on MGF values computed by simulation. Therefore, should the simulation and curve fitting be repeated, or a different number of samples be used, the coefficients may vary. However, the variation should naturally be within a small range.

	bT = 0.001	0.01	0.1	0.2	0.5	1	2	5	10
a_1	-9.003E-2	1.397E-1	-1.268E+0	1.516E+0	3.208E-1	1.200E-5	-2.455E-1	-5.769E-1	2.056E+0
a_2	-3.336E-4	-8.202E-2	-7.960E-1	-9.439E-1	1.191E-1	1.788E-1	1.190E-2	-6.415E-1	1.627E+0
a_3	6.108E-1	6.153E-3	4.291E+0	-3.231E+0	-7.277E-1	1.240E-1	7.558E-1	1.226E+0	-4.673E+0
a_4	7.092E-4	2.071E-1	1.254E+0	2.295E+0	-9.730E-2	-3.319E-1	6.554E-2	1.401E+0	-3.696E+0
a_5	5.536E-1	1.168E+0	-3.975E+0	3.734E+0	1.587E+0	4.382E-1	-4.512E-1	-7.554E-1	4.129E+0
a_6	-5.968E-4	-2.082E-1	-5.194E-1	-2.125E+0	-1.119E-1	1.844E-1	-1.648E-1	-1.163E+0	3.266E+0
a_7	-8.928E-2	-4.257E-1	2.584E+0	-1.481E+0	-7.153E-1	-1.565E-1	2.685E-1	2.392E-1	-1.795E+0
a_8	2.674E-4	1.111E-1	-5.723E-2	9.920E-1	1.736E-1	1.426E-2	1.491E-1	4.919E-1	-1.491E+0
a_9	1.544E-2	1.233E-1	-8.603E-1	3.414E-1	1.913E-1	4.221E-2	-6.984E-2	-2.620E-2	4.352E-1
a_{10}	-7.109E-5	-3.505E-2	9.509E-2	-2.530E-1	-6.093E-2	-1.569E-2	-4.621E-2	-1.148E-1	3.930E-1
a_{11}	-1.985E-3	-2.330E-2	1.643E-1	-4.869E-2	-3.126E-2	-7.037E-3	1.105E-2	-7.534E-4	-6.089E-2
a_{12}	1.184E-5	6.858E-3	-2.679E-2	3.929E-2	1.122E-2	3.587E-3	7.966E-3	1.656E-2	-6.267E-2
a_{13}	1.750E-4	2.793E-3	-1.884E-2	4.394E-3	3.197E-3	7.315E-4	-1.102E-3	5.176E-4	4.987E-3
a_{14}	-1.246E-6	-8.379E-4	3.697E-3	-3.828E-3	-1.224E-3	-4.291E-4	-8.335E-4	-1.511E-3	6.168E-3
a_{15}	-9.928E-6	-2.040E-4	1.284E-3	-2.448E-4	-2.002E-4	-4.636E-5	6.766E-5	-5.738E-5	-2.291E-4
a_{16}	8.049E-8	6.199E-5	-2.808E-4	2.292E-4	7.963E-5	2.939E-5	5.270E-5	8.493E-5	-3.675E-4
a_{17}	3.254E-7	8.241E-6	-4.812E-5	7.698E-6	7.030E-6	1.642E-6	-2.338E-6	2.833E-6	5.108E-6
a_{18}	-2.913E-9	-2.529E-6	1.128E-5	-7.725E-6	-2.865E-6	-1.092E-6	-1.856E-6	-2.681E-6	1.218E-5
a_{19}	-4.672E-9	-1.407E-7	7.642E-7	-1.046E-7	-1.061E-7	-2.495E-8	3.479E-8	-5.441E-8	-3.410E-8
a_{20}	4.522E-11	4.356E-8	-1.878E-7	1.123E-7	4.393E-8	1.712E-8	2.798E-8	3.635E-8	-1.725E-7

Table 1: Coefficients of polynomial curve fitting of ¡(s)

4. Conclusion

We have demonstrated a new method of statistically characterizing filtered phase noise in optical heterodyne receivers that use matched filter demodulators. The method is based on least squares fitting of the log MGF of the filtered phase noise random variable using a power series. The accuracy of the new method has been verified through plots of the fitting error which was clearly shown to be negligibly small.

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