# Statistical Characterization of Filtered Phase Noise in Optical Receivers

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***Abstract*—This letter presents an accurate numerical approach to statistically characterize filtered phase noise, usually encountered in the analysis of heterodyne optical fiber communication receivers. Filtered phase noise is generated by simulation. Generated samples are used to estimate the probability density functions and moment generating functions of phase noise random variables. The proposed approach is valid for large as well as small phase noise. Furthermore, it comes close to providing analytical expressions for the computed statistical measures.**

***Index Terms*—Filtered phase noise, heterodyne detection, moment generating function, probability density function.**

## I. Introduction

IT IS WELL KNOWN that semiconductor laser phase noise can be a major source of performance degradation in heterodyne optical receivers [1]–[4]. A vast majority of previous and current research on system performance in the presence of phase noise assumes a matched filter model for the demodulator. This model has not been proved to be optimal in terms of error rate performance. A good reason for the lack of an optimal heterodyne optical receiver that takes phase noise into account is that filtered phase noise is very difficult to characterize statistically.

Even with the matched filter model there are no analytical or accurate numerical statistical characterizations of filtered phase noise. However, attempts at finding various moments of decision variables involving filtered phase noise can be found in the literature. Examples include [5] and [6]. The objective of the present paper—as will be illustrated later—is to provide probability density functions (pdfs) and moment generating functions (mgfs) of filtered phase noise.

Several phase noise-based quantities contribute to decision variable expressions that appear in matched filter analysis. A frequently encountered quantity is [4], [7]–[10]

$$\eta = \frac{1}{T}\int_0^T e^{j\theta(t)}dt \tag{1}$$

where $T$ is a time interval (usually related to symbol or chip duration) and $\theta(t)$ is a Wiener–Levy process representing laser phase noise with linewidth $\beta$. The phase noise process is usually written in the form

$$\theta(t) = 2\pi\int_0^t \varphi(\zeta)d\zeta \tag{2}$$

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where $\varphi(\zeta)$ is a wide-sense stationary zero-mean random process with flat power spectral density equal to $\beta/2\pi$ [4].

Most existing attempts to model filtered phase noise assume a small linewidth compared to receiver IF bandwidth, or equivalently, a small (symbol or chip) linewidth-duration product $\beta T$ [4], [7], [8]. However, in many interesting cases this assumption does not apply and its incorporation into error rate evaluations leads to invalid results.

In this letter, we propose a numerical/simulation approach that leads to accurate pdfs and mgfs of the random variable $|\eta|^2$. The mgf curves are particularly important because they can sometimes be used to directly calculate error rates without the need for pdf integration.

The remainder of this paper is organized as follows. Section II presents some analytical expressions for the first- and second-order moments of the random variable to be studied. These are used to verify the accuracy of the numerical approach. Sections III and IV outline the simulation process and contain the results of this paper in the form of pdf and mgf plots and a table of mgf least squares polynomial curve fitting. Conclusions are summarized in Section V.

## II. Basic Moments of Filtered Phase Noise

Let's consider the random variable $\eta$ in (1). The expected value of $\eta$ can be easily found as follows:

$$\mu_\eta = E[\eta] = \frac{1}{T}\int_0^T e^{-\pi\beta t}dt = \frac{1-e^{-D}}{D} \tag{3}$$

where

$$D = \pi\beta T. \tag{4}$$

Note that the fact that $\theta(t)$ is zero-mean gaussian with variance equal to $2\pi\beta T$ has been used. Now, let's define

$$U = |\eta|^2. \tag{5}$$

This quantity is usually encountered in the decision variables of noncoherent demodulators. It has been shown in [4] that

$$\mu_U = E[U] = \frac{2}{D}\left[1-\mu_\eta\right] \tag{6}$$

and

$$E\left[U^2\right] = \frac{783-540D+144D^2-784e^{-D}-240De^{-D}+e^{-4D}}{18D^4}. \tag{7}$$

Using a small phase noise approximation, the mgf of $U$ was expressed in [4] in the form

$$\Psi_U(s) = e^s \sqrt{\frac{2\sqrt{Ds}}{\sinh(2\sqrt{Ds})}}. \tag{8}$$

## III. Simulation Procedure

Our focus will be on finding accurate statistical characterizations of $U$. Note that (1) can rewritten in the form

$$\eta = \sum_{k=1}^{N} \frac{1}{T} \int_{(k-1)T/N}^{kT/N} e^{j\theta(t)dt}. \tag{9}$$

When $N$ is sufficiently large the integrand can be assumed to be constant over the integration interval, leading to the approximation

$$\eta \approx \frac{1}{N} \sum_{k=1}^{N} e^{j\theta_k} = \frac{1}{N} \sum_{k=1}^{N} [\cos(\theta_k) + j\sin(\theta_k)] = \eta_c + j\eta_s \tag{10}$$

where $\theta_k = \theta((k-1)T/N)$. From the properties of $\theta(t)$ it can be easily deducted that $\theta_k$ is zero-mean Gaussian with variance equal to $2(k-1)D/N$. It can be seen from (2) that

$$\theta_{k+1} = \theta_k + 2\pi \int_{k-1)T/N}^{kT/N} \varphi(\zeta)d\zeta = \theta_k + \phi_k \tag{11}$$

where $\phi_k$ is zero-mean gaussian with variance equal to $2D/N$. Note that while the set of random variables $\{\phi_k\}$ are statistically independent, the set $\{\theta_k\}$ are not.

The pdfs and mgfs of $U$ have been estimated as follows.

1) To make the simulation accurate enough, $N = 1000$ was used.
2) A new independent $\phi_k$ was generated for every $k = 2, 3, \ldots, N$. Note that $\theta_1 = 0$.
3) A sample of $U$ was generated according to

$$U = \eta_c^2 + \eta_s^2. \tag{12}$$

4) Steps 2 and 3 were repeated for a large number ($K \geq 10^5$) of times.
5) Histograms were found from step 4, then there areas were normalized to unity so that they give pdf estimates.
6) The mgfs were calculated according to

$$\Psi_U(s) = E\left[e^{sU}\right] \tag{13}$$

by calculating of the mean of the product of $s$ and values of $U$ in step 4.

7) Least squares polynomial curve fittings of $\Upsilon_U(s) = \ln \Psi_U(s)$ were performed. Polynomials of degree 9 were found to be accurate enough. The result is an approximation in the form

$$\Upsilon_U(s) = \sum_{n=0}^{9} a_n s^n. \tag{14}$$

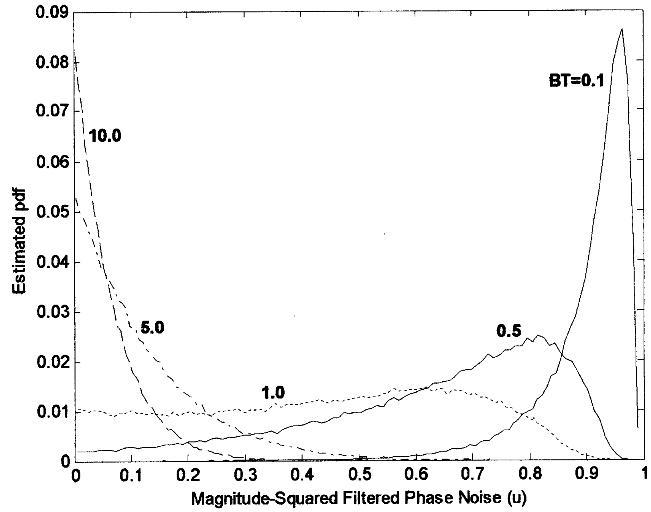


Fig. 1. Estimated pdf of the magnitude-squared filtered phase noise random variable $U$. The linewidth-duration product ranges from 0.1–10.

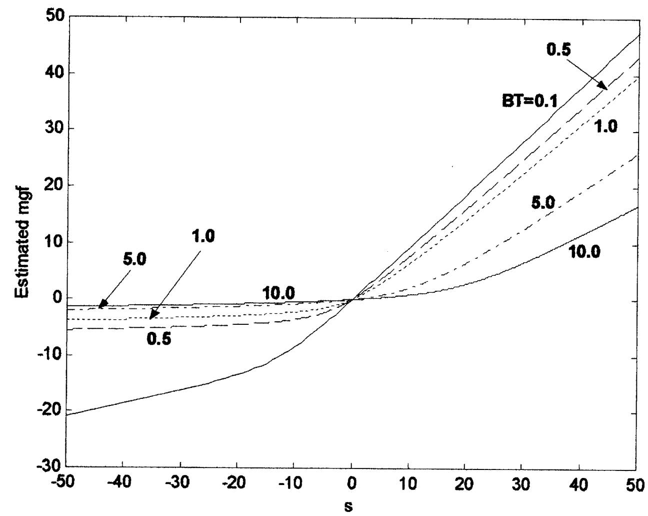


Fig. 2. Estimated mgf of the magnitude-squared filtered phase noise random variable $U$. The linewidth-duration product ranges from 0.1 to 10.

## IV. Results

Fig. 1 shows estimates of the pdf of $U$ for a wide range of $\beta T$ values. Equations (6) and (7) has been verified for all these cases to be true. It can be anticipated from this plot that $U$ can seriously affect system performance when $\beta T \geq 1$.

Fig. 2 shows estimates of the mgf of $U$ for the same range of $\beta T$ values. Table I gives the coefficients of polynomial curve fitting of $\Upsilon_U(s)$.

Fig. 3 shows the least squares curve fitting error. Linewidth-duration product values ($\beta T = 0.1$, $\beta T = 1.0$, $\beta T = 10.0$) are chosen to cover a wide range of phase noise levels. These error curves demonstrate that polynomial fitting of $\Upsilon_U(s)$ can lead to a reasonable representation of the mgf. Fig. 4 compares our mgf estimates to those of [4] according to (8).

As can be seen, the small phase noise approximation leads to quite large representation errors when phase noise is not actually small.

TABLE I
COEFFICIENTS OF POLYNOMIAL CURVE FITTING OF $\Upsilon_U(s)$

| | $\beta T = 0.1$ | 0.2 | 0.5 | 1.0 | 2.0 | 5.0 | 10.0 |
|---|---|---|---|---|---|---|---|
| $a_9$ | 1.03314E-14 | 2.55216E-14 | 1.53833E-14 | 1.83419E-15 | -9.1494E-15 | -9.8820E-15 | 1.55572E-15 |
| $a_8$ | 3.37406E-13 | -2.6926E-13 | -8.4732E-13 | -8.518E-13 | -5.9836E-13 | 2.59492E-14 | 2.22215E-13 |
| $a_7$ | -7.5197E-11 | -1.6465E-10 | -9.6351E-11 | -1.0855E-11 | 5.8651E-11 | 6.74181E-11 | -3.9268E-12 |
| $a_6$ | -1.6679E-09 | 1.95552E-09 | 5.15376E-09 | 5.12915E-09 | 3.67028E-09 | 6.02783E-11 | -1.2755E-09 |
| $a_5$ | 2.1363E-07 | 3.90925E-07 | 2.17625E-07 | 2.15416E-08 | -1.3881E-07 | -1.762E-07 | -1.9071E-08 |
| $a_4$ | 1.88305E-06 | -5.8897E-06 | -1.1894E-05 | -1.1647E-05 | -8.6564E-06 | -1.2693E-06 | 2.20613E-06 |
| $a_3$ | -0.00031323 | -0.0004328 | -0.00021744 | 1.2378E-05 | 0.000157853 | 0.00023492 | 9.33904E-05 |
| $a_2$ | 0.005709705 | 0.013799464 | 0.018119152 | 0.017364424 | 0.014385288 | 0.007207856 | 0.001844337 |
| $a_1$ | 0.910565987 | 0.785351388 | 0.577353476 | 0.431088591 | 0.293758381 | 0.125240126 | 0.057355261 |
| $a_0$ | -0.10243852 | -0.0822314 | 0.179314733 | 0.242034993 | -0.13200268 | 0.04530168 - | 0.01201977 |

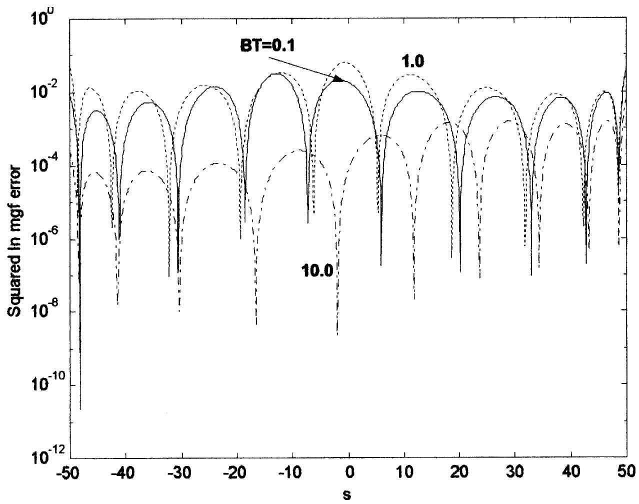


Fig. 3. Least squares curve fitting error of the natural logarithm of the mgf.

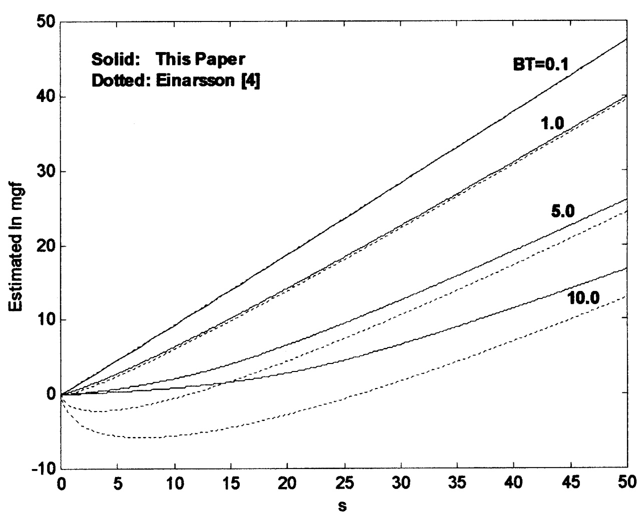


Fig. 4. Comparison of mgf estimates in this letter and those in [4].

## V. CONCLUSIONS

An accurate numerical statistical characterization of filtered phase noise has been presented. The importance of our results is twofold. First, they can be used in error rate analysis of heterodyne optical receivers with noncoherent demodulators. This can be seen by noting that in many occasions the error rate expression can be written in the form of an mgf. Second, they provide a general statistical representation that is useful for large, as well as small phase noise.

## REFERENCES


[1] G. Nicholson, "Probability of error for optical heterodyne DPSK system with quantum phase noise," *Electron. Lett.*, vol. 20, pp. 1005–1007, 1984.

[2] C. H. Henry, "Phase noise in semiconductor lasers," *J. Lightwave Technol.*, vol. LT-4, pp. 298–311, Mar. 1986.

[3] K. Kikuchi, T. Okoshi, M. Nagamtsu, and N. Hennmi, "Degradation of bit error rate in coherent optical communication due to spectral spread of the transmitter and local oscillator," *J. Lightwave Technol.*, vol. LT-2, pp. 1024–1033, Dec. 1984.

[4] G. Einarsson, *Principles of Lightwave Communications*. New York: Wiley, 1996.

[5] L. G. Kazovsky and O. K. Tonguz, "ASK an FSK coherent lightwave systems: A simplified approximate analysis," *J. Lightwave Technol.*, vol. 8, pp. 338–352, Mar. 1990.

[6] O. K. Tonguz and L. G. Kazovsky, "Theory of direct detection lightwave receivers using optical amplifiers," *J. Lightwave Technol.*, vol. 9, pp. 174–181, Feb. 1991.

[7] G. J. Foschini and G. Vannucci, "Characterising filtered lightwaves corrupted by phase noise," *IEEE Trans. Inform. Theory*, vol. 34, pp. 1437–1448, Nov. 1988.

[8] M. Stefanović, D. Milić, and N. Stojanović, "Evaluation of optimal bandwidth in optical FSK system influenced by laser phase noise," *J. Lightwave Technol.*, vol. 16, pp. 772–777, May 1998.

[9] G. J. Foschini and L. J. Greenstein, "Noncoherent detection of coherent lightwave signals corrupted by phase noise," *IEEE Trans. Commun.*, vol. 36, pp. 306–314, Mar. 1988.

[10] J. Fan, G. Jacobsen, and L. Kazovsky, "Preamplifier ASK system performance with incomplete ASK modulation: Influence of ASE and laser phase noise," *J. Lightwave Technol.*, vol. 13, pp. 302–311, Feb. 1995.