V. Complex Matrices

Definition V-1

The complex conjugate of matrix A is obtained by conjugating element of, i.e.,

$$A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \Rightarrow$$
$$A^*=\begin{bmatrix} a_{11}^* & a_{12}^* & \cdots & a_{1n}^* \\ a_{21}^* & a_{22}^* & \cdots & a_{2n}^* \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}^* & a_{m2}^* & \cdots & a_{mn}^* \end{bmatrix} \quad \text{(V.1)}$$

$$h=h_a+ih_b \Rightarrow h^*=h_a-ih_b \Rightarrow |h|^2=h_a^2+h_b^2=\left|h^*\right|^2$$

$$h=2-3i \Rightarrow h^*=2+3i$$

Definition V-2

The Hermitian transpose of matrix A is obtained by transposing the complex conjugate of A, or by complex conjugating the transpose of A, and is given by

$$\begin{aligned} A^H &= \left(A^*\right)^T \\ &= \left(A^T\right)^* \end{aligned} \quad \text{(V.2)}$$

$$A=\begin{bmatrix} 1-2i & 3+3i \\ 0-2i & 5+0i \end{bmatrix} \Rightarrow A^*=\begin{bmatrix} 1+2i & 3-3i \\ 2i & 5 \end{bmatrix} \Rightarrow A^H=\begin{bmatrix} 1+2i & 2i \\ 3-3i & 5 \end{bmatrix}$$

$$\underline{a}\cdot\underline{b}=\underline{a}^H\underline{b}=\left(\underline{b}^H\underline{a}\right)^*$$

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$$\underline{a}=\begin{bmatrix}1-1i\\2+4i\\6i\end{bmatrix},\underline{b}=\begin{bmatrix}1+3i\\2\\2-i\end{bmatrix}\Rightarrow \underline{a}\cdot\underline{b}=\begin{bmatrix}1+1i & 2-4i & -6i\end{bmatrix}\begin{bmatrix}1+3i\\2\\2-i\end{bmatrix}=(1+1i)(1+3i)+2(2-4i)-6i(2-i)$$

$$=1+3i+1i-3+4-8i-12i-6$$
$$=-4-16i$$

$\|\underline{x}\|^2=\underline{x}^H\underline{x}=\sum_{i=1}^{n}x_i^*x_i=\sum_{i=1}^{n}|x_i|^2$. This is the inner product.

$$|\alpha|^2=\alpha^*\alpha$$
$$|\alpha|=\sqrt{\alpha^*\alpha}$$

$$\underline{a}=\begin{bmatrix}1-1i\\2+4i\\6i\end{bmatrix}\Rightarrow\|\underline{a}\|^2=2+20+36=58$$

If $A^H=A$, then A is Hermitian and $\underline{z}^H A\underline{z}$ is real.

$$A=\begin{bmatrix}2 & 3+3i\\3-3i & 5\end{bmatrix}$$

Every eigenvalue of a Hermitian matrix is real.

The eigenvectors of a Hermitian matrix are orthogonal.

A unitary matrix U is a (complex) square matrix that has orthonormal columns. $U^HU=I$

$U=\frac{1}{\sqrt{3}}\begin{bmatrix}1 & 1-i\\1+i & -1\end{bmatrix}$ is unitary.

$$1(1+i)+(1+i)(-1)=0$$

$$\left\|\begin{bmatrix}1\\1+i\end{bmatrix}\right\|^2=1(1)+(1+i)(1-i)=1+1+1=3 \qquad \left\|\begin{bmatrix}1-i\\-1\end{bmatrix}\right\|^2=3 \qquad \left\|\begin{bmatrix}1-i\\-1\end{bmatrix}\right\|=\sqrt{3}$$

$$\left\|\begin{bmatrix}1\\1+i\end{bmatrix}\right\|=\sqrt{3}$$