TABLE OF CONTENTS

SYLLABUS

Course Catalog

3 Credit hours (3 h lectures). Discrete and continuous time systems: classifications, convolution and impulse response. Orthogonal expansions and Fourier series. Fourier transform. Laplace transform. Z-transform. System function. Computer applications.

Textbook

Signals, Systems, and Transforms, Charles L. Philips, Fourth Edition, Printice Hall , ISBN 0-13-206742-0.

References

1. Roberts & Gasbel. *Linear Signals & Systems*. 3^{nd} ed.

Instructor

Instructor: Dr. Mohammad M. Banat

Email Address: banat@just.edu.jo

Prerequisites

Prerequisites by topic Circuits, Linear Algebra

Prerequisites by course EE 210, EE 240

Prerequisite for EE 360

Topics Covered

Week	Topics	Chepters in Text
1-4	Continuous Time Signals and Systems	3
5	Fourier Series	4
6-8	Fourier Transform and Applications	5-6
9	Laplace Transform	7
10-13	Discrete Time Signals and Systems	9-10
14	z Transform	11

Objectives and Outcomes

Objectives	Outcomes
1. Classifying signals and systems as represented by their mathematical models [1]	1.1. Defining basic operations such as time scale, time shift, time reverse, and combinations of these operations for signals. [1] 1.2. Learning properties and classification of continuous-time as well as discrete-time signals [1] 1.3. Determine continuous- as well as discrete-time system characteristics (e.g., causality, linearity, time-invariance, etc.) [1]
2. Analyzing both continuous and discrete linear time-invariant systems in the time domain [1]	2.1. Determining & applying differential equation models for linear time-invariant systems and circuits (continuous- and discrete-time) [1] 2.2. Using graphical and analytical methods to compute a convolution (continuous time and discrete-time) [1]
3. Applying the Fourier representation of signals and systems to analyze continuous linear systems in the frequency domain [1]	3.1. Calculating Fourier series expansions for periodic continuous-time signals and plot line spectra [1] 3.2. Implementing the forward and inverse Fourier transforms to analyze signals and systems [1] 3.3. Obtaining frequency response of a system using Fourier Transform [1] 3.4. Using Fourier transform methods for analysis of linear systems [1]
4. Implementing the Laplace representation of signals and systems in analyzing linear systems[1]	4.1. Performing Laplace transform for signals [1] 4.2. Identifying system transfer function [1] 4.3. Using Laplace transform methods for analysis of continuous-time linear systems [1]
5. Applying the discrete Fourier representation and Z-transform of signals and systems to analyze continuous linear systems in the frequency domain [1]	5.1. Calculating the discrete-time Fourier transform of signals [1 5.2. Identifying the Z-transform for discrete-time signals and plotting its region of convergence [1] 5.3. Differentiating between bilateral and unilateral Z-transforms [1] 5.4. Using the forward Z-transform and inverse Z-transform to analyze signals and systems [1]

Evaluation

Assessment Tool	**Expected Due Date**	**Weight**
Exam 1		20%
Exam 2		20%
Class Work		20%
Final Exam		60%

Contribution of Course to Meeting the Professional Component

The course contributes to equip students with basic knowledge and skills in applied probability and random processes.

Relationship to Program Outcomes (%)

A	B	C	D	E	F	G	H	I	J	K	L

Relationship to Electrical Engineering Program Objectives

PEO1	PEO2	PEO3	PEO 4	PEO 5

I. INTRODUCTION

Engineers must model two distinct physical phenomena. The first is physical systems, which can be modeled by mathematical equations. For example, continuous-time, or analog, systems (systems that contain no sampling) can be modeled by ordinary differential equations with constant coefficients. A second physical phenomenon to be modeled is called a signal. Physical signals are modeled by mathematical functions. One example of a physical signal is the voltage that is applied to the speaker in a radio. Another example is the temperature at a designated point in a particular room. This signal is a function of time, since the temperature varies with time. We can express this temperature as

$$\text{Temperature at a point} = \theta(t) \tag{I.1}$$

where $\theta(t)$ has the units of, for example, degrees Celsius. To be more precise in this example, the temperature in a room is a function of time and of space.

$$\text{Temperature at a point} = \theta(x, y, z, t) \tag{I.2}$$

where the point in a room is identified by the three space coordinates x, y, and z. We limit signals to having one independent variable. In general, this independent variable will be time t.

Signals are divided into two natural categories. The first category to be considered is continuous-time signals, or simply, continuous signals. The second category for signals is discrete-time signals, or simply, discrete signals.

A continuous-time signal is defined for all values of time. A continuous-time signal is also called an analog signal. A continuous-time system is a system in which only continuous-time signals appear. There are two types of continuous time signals. A continuous-time signal $x(t)$ can be a continuous-amplitude signal, for which the time-varying amplitude can assume any value. A continuous-time signal may also be a discrete-amplitude signal, which can assume only certain defined amplitudes. An example of a discrete-amplitude continuous-time signal is the output of a digital-to-analog converter. For example, if the binary signal into the digital-to-analog converter is represented by eight bits, the output-signal amplitude can assume only $2^8 = 256$ different values.

A discrete signal is defined at only certain instants of time. For example, suppose that a signal $f(t)$ is to be processed by a digital computer. Since a computer can operate on only a number and not a continuum, the continuous signal must be converted into a sequence of numbers by sampling. This sequence of numbers is called a discrete-time signal. Like continuous-time signals, discrete-time signals can be either continuous amplitude or discrete amplitude. A discrete-time system is a system in which only discrete-time signals appear.

I.1. Transformations of Continuous-Time Signals

I.1.A. TIME TRANSFORMATIONS

Time Reversal

$$y(t) = x(-t) \tag{I.3}$$

The time reversal operation is shown in Figure I.1.

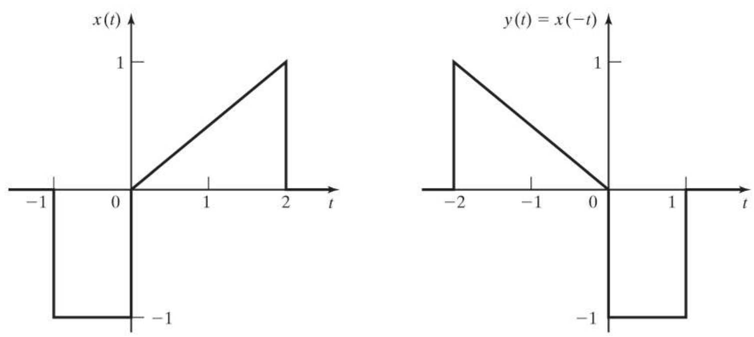


Figure I.1: Time reversal

Time Scaling

$$y(t) = x(at), \quad a \in \mathbb{R} \tag{I.4}$$

The time scaling operation is shown in Figure I.2.

Time Shifting

$$y(t) = x(t - t_0) \tag{I.5}$$

General Transformation

$$y(t) = x(at + b) \tag{I.6}$$

Let

$$\tau = at + b \tag{I.7}$$

Then

$$t = \frac{\tau}{a} - \frac{b}{a} \tag{I.8}$$

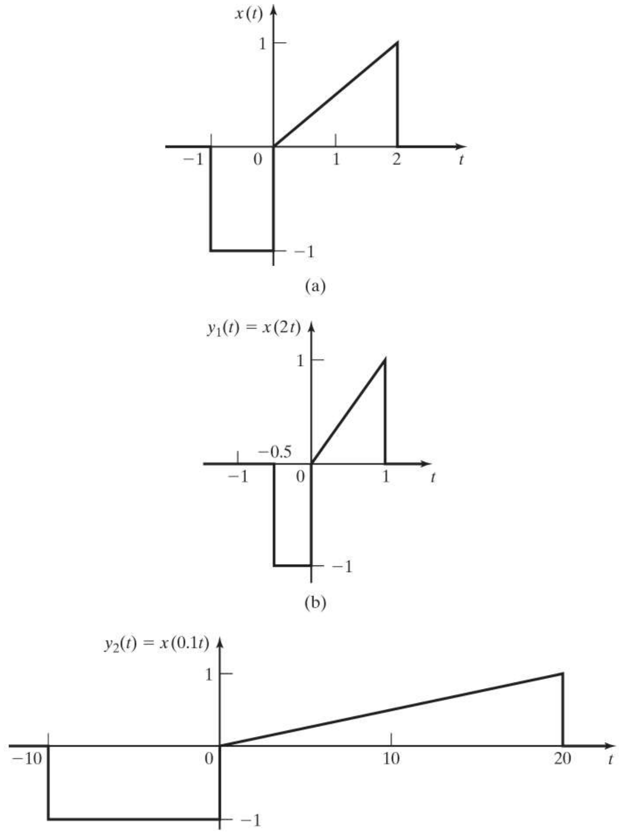


Figure I.2: Time scaling

Example

Let

$$y(t) = x\left(1 - \frac{t}{2}\right)$$

Then

$$t = 2 - 2\tau$$

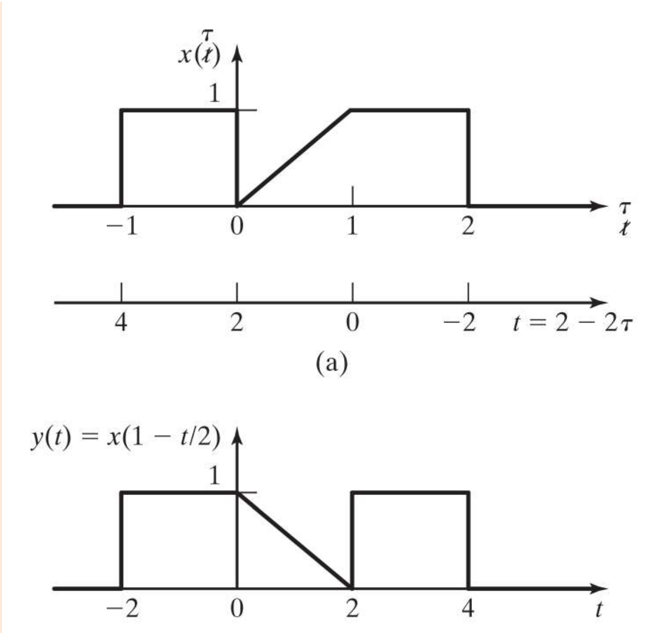


Figure I.3: Time transformation example

I.1.B. Amplitude Transformations

$$y(t) = ax(t) + b \tag{I.9}$$

where a and b are constants.

Example

Let

$$y(t) = 3x(t) - 1$$

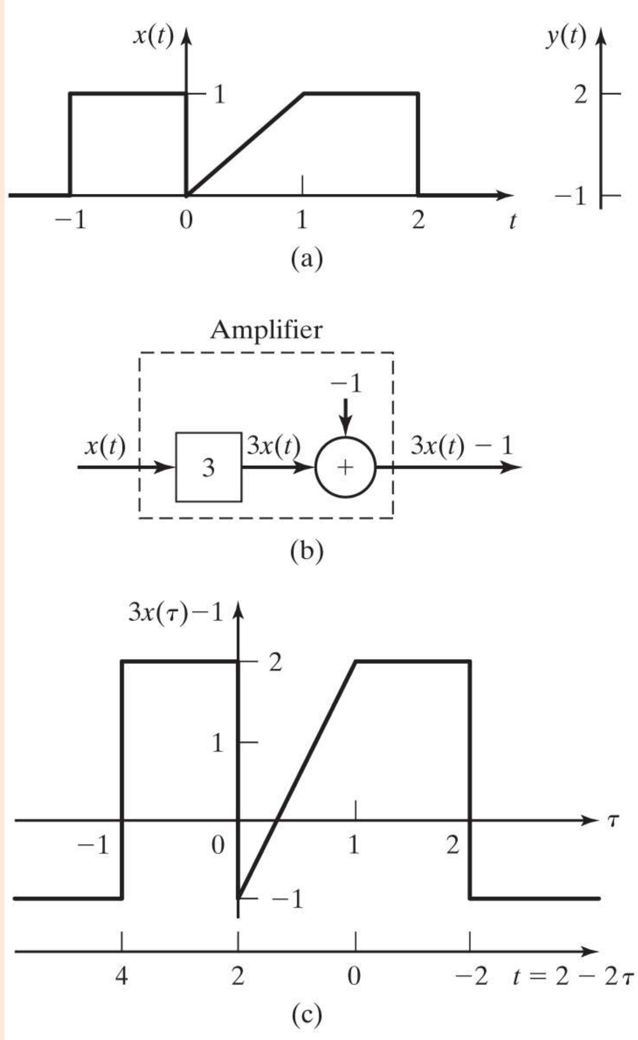


Figure I.4: Amplitude transformation example

Example

Next we consider the signal

$$y(t) = 3x\left(1 - \frac{t}{2}\right) - 1$$

which has time transformation and amplitude transformation. To plot this transformed signal, we first transform the amplitude axis, as shown below. The t-axis is redrawn to facilitate the time transformation.

$$\tau = 1 - \frac{t}{2} \Rightarrow t = 2 - 2\tau$$

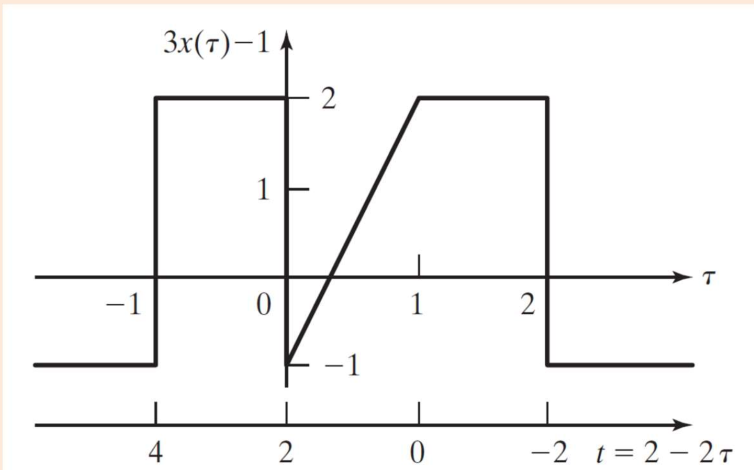


Figure I.5: Time and amplitude transformation

The signal is then plotted on the t -axis, as shown below.

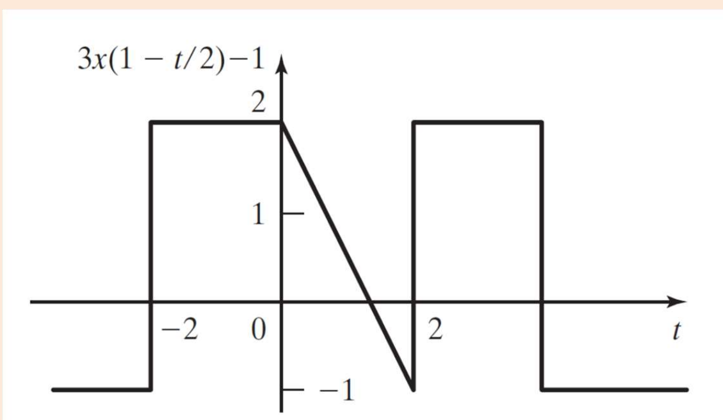


Figure I.6: Time and amplitude transformation

I.2. Signal Characteristics

I.2.A. Even and Odd Signals

By definition, the function (signal) is even if

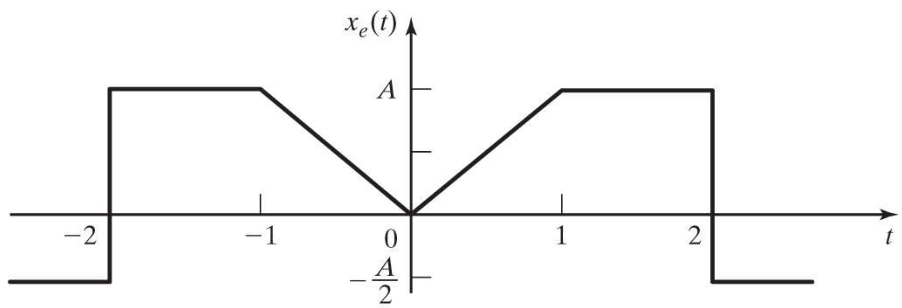
$$x_e(t) = x_e(-t) \tag{I.10}$$

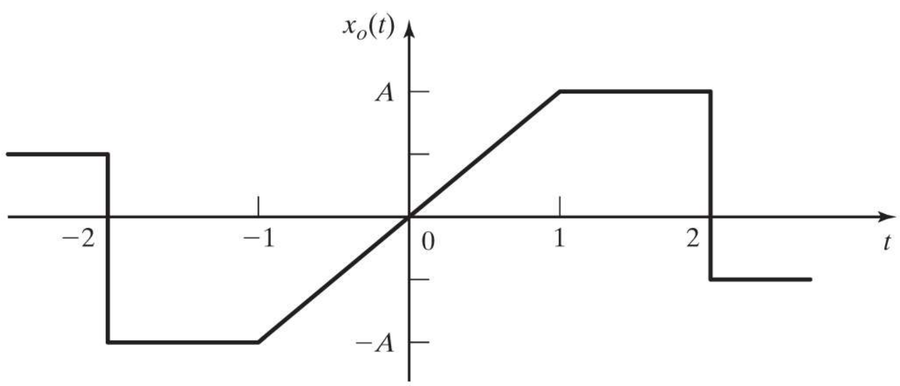
An even function has symmetry with respect to the vertical axis; the signal for $t < 0$ is the mirror image of the signal for $t > 0$. The function $x(t) = \cos(\omega_0 t)$ is an even function because $\cos(\omega_0 t) = \cos(-\omega_0 t)$.

By definition, a function is odd if

$$x_o(t) = -x_o(-t) \tag{I.11}$$

An odd function has symmetry with respect to the origin. The function $x(t) = \sin(\omega_0 t)$ is odd because $\sin(\omega_0 t) = -\sin(-\omega_0 t)$.





Any signal can be expressed as the sum of an even part and an odd part; that is,

$$x(t) = x_e(t) + x_o(t) \tag{I.12}$$

Note that

$$x(-t) = x_e(t) - x_o(t) \quad \text{(I.13)}$$

Adding (I.12) and (I.13) and dividing by 2 yields

$$x_e(t) = \frac{1}{2}\left[x(t) + x(-t)\right] \quad \text{(I.14)}$$

Subtracting (I.13) from (I.12) and dividing by 2 yields

$$x_o(t) = \frac{1}{2}\left[x(t) - x(-t)\right] \quad \text{(I.15)}$$

The average value A_x of a signal $x(t)$ is defined as

$$A_x = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)dt \quad \text{(I.16)}$$

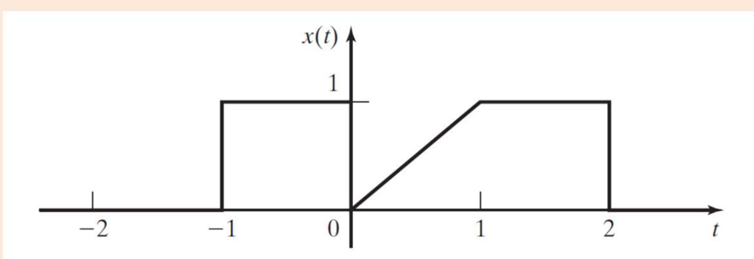
The average value of a signal is contained in its even function, since the average value of a bounded odd function is zero.

Even and odd functions have the following properties:

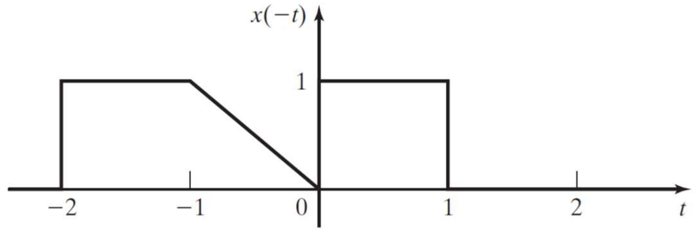
1. The sum of two even functions is even.
2. The sum of two odd functions is odd.
3. The sum of an even function and an odd function is neither even nor odd.
4. The product of two even functions is even.
5. The product of two odd functions is even.
6. The product of an even function and an odd function is odd.

Example

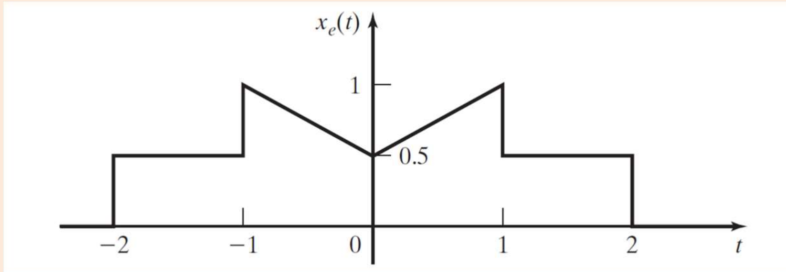
Consider the signal $x(t)$:



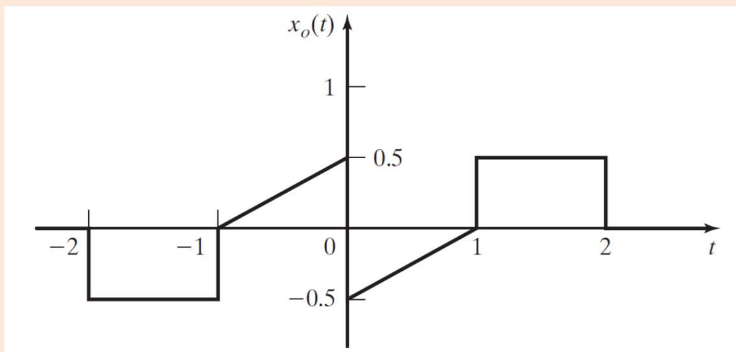
Following is the time-reversed signal $x(-t)$:


x(−t)
1
−2
−1
0
1
2
t

The two signals are added and scaled in amplitude by 0.5 to yield the even signal $x_e(t)$:



Next, $x(-t)$ is subtracted from $x(t)$, and the result is amplitude scaled by 0.5 to yield the odd signal $x_o(t)$:



I.2.B. Periodic Signals

A continuous-time signal $x(t)$ is periodic if for all t and positive T we have

$$x(t+T) = x(t) \tag{I.17}$$

A signal that is not periodic is said to be aperiodic.

Constant T is the period of $x(t)$. Replacing t with $t+T$ in (I.17), we get

$$\begin{aligned} x(t+2T) &= x(t+T) \\ &= x(t) \end{aligned} \tag{I.18}$$

We can repeat the above step until we get for any integer n

$$x(t+nT) = x(t) \tag{I.19}$$

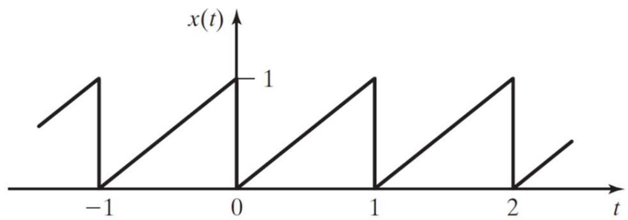
Hence, a periodic signal with period T is also periodic with period nT, which means that a periodic signal has infinitely many periods that are all integer multiples of T. Since T is the smallest period of $x(t)$, it is called the fundamental period. Symbol T_0 is often used to denote the fundamental period.

T_0 is usually measured in seconds. The fundamental frequency (measured in Hz) is given by

$$f_0 = \frac{1}{T_0} \quad \text{(I.20)}$$

The fundamental angular frequency (measured in rad/s) is given by

$$\begin{aligned}\omega_0 &= 2\pi f_0 \\ &= \frac{2\pi}{T_0}\end{aligned} \quad \text{(I.21)}$$



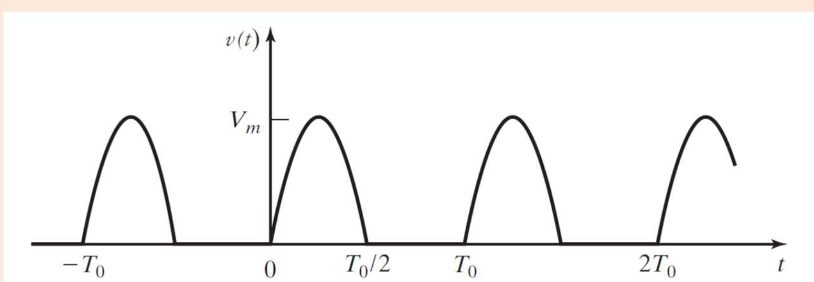
Other periodic signals are:

- $x_c(t) = \cos\omega_0 t$.
- $x_s(t) = \sin\omega_0 t$.
- $x(t) = \text{constant}$. Period is undefined.

Example

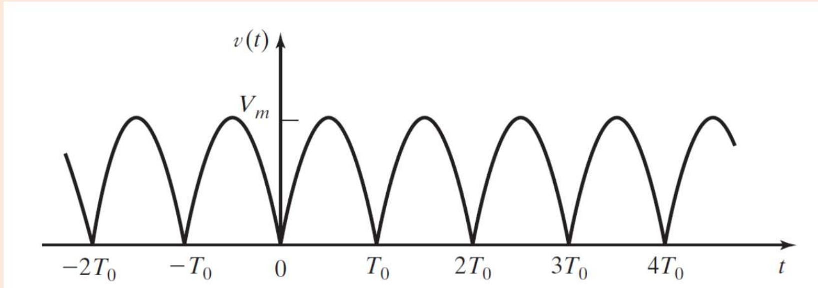
Power supplies that convert an ac voltage (sinusoidal voltage) into a dc voltage (constant voltage) are required in almost all electronic equipment that doesn't use batteries.

Following is a half-wave rectified signal:



This signal is generated from a sinusoidal signal by replacing the negative half cycles of the sinusoid with a voltage of zero. The positive half cycles are unchanged.

Following is a full-wave rectified signal:



This signal is generated from a sinusoidal signal by the amplitude reversal of each negative half cycle. The positive half cycles are unchanged. Note that the period of this signal is one-half that of the sinusoid and, hence, one-half that of the half-wave rectified signal.

The sum of continuous-time periodic signals is periodic if and only if the ratios of the periods of the individual signals are ratios of integers. If a sum of N periodic signals is periodic, the fundamental period can be found as follows

1. Convert each period ratio T_{01}/T_{0n} for $n = 2, \cdots, N$ to a ratio of integers, where T_{01} is the period of the first signal considered and T_{0n} is the period of one of the other $N-1$ signals. If one or more of these ratios is not rational, then the sum of signals is not periodic.
2. Eliminate common factors from the numerator and denominator of each ratio of integers.
3. The fundamental period of the sum of signals is $T_0 = k_0 T_{01}$ where k_0 is the least common multiple of the denominators of the individual ratios of integers.

Example

Consider the signals

$$x_1(t) = \cos 3.5t, \quad T_{01} = \frac{2\pi}{3.5}$$

$$x_2(t) = \sin 2t, \quad T_{02} = \frac{2\pi}{2}$$

$$x_3(t) = 2 = \cos \frac{7}{6} t, \quad T_{03} = \frac{2\pi}{7/6}$$

Let

$$v(t) = x_1(t) + x_2(t) + x_3(t)$$

Then

$$N = 3$$

$$\frac{T_{01}}{T_{02}} = \frac{4}{7} \text{ (ratio of integers)}, \quad \frac{T_{01}}{T_{03}} = \frac{7}{21} \text{ (ratio of integers)}$$

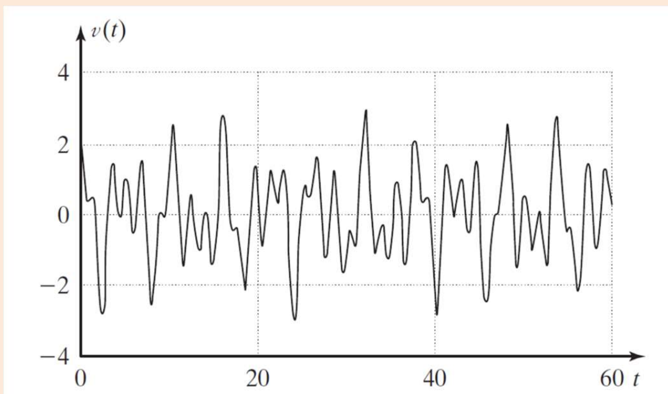
$v(t)$ is periodic.

$$\frac{T_{01}}{T_{03}} = \frac{1}{3}$$

Least common multiple of denominators is $k_0 = 21$.

Fundamental period of $v(t)$ is

$$\begin{aligned} T_0 &= k_0 T_{01} \\ &= 21 \times \frac{2\pi}{3.5} \\ &= 12\pi \end{aligned}$$

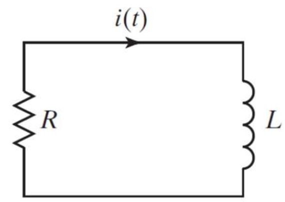


I.3. Sinusoidal Signals

Continuous-time systems can be modeled using ordinary linear differential equations with constant coefficients. A signal that appears often in these models is one whose time rate of change is directly proportional to the signal itself. An example of this type of signal is the differential equation

$$\frac{d}{dt} x(t) = a x(t) \tag{I.22}$$

where a is constant. The solution of this equation is the exponential function $x(t) = x(0)e^{at}$ for $t \geq 0$. An example is the current in an RL circuit.



$$L\frac{d}{dt}i(t)+\frac{R}{L}i(t)=0$$

$$\frac{d}{dt}i(t)=-\frac{R}{L}i(t)$$

Thus,

$$a=-R/L$$

$$i(t)=i(0)e^{-\frac{R}{L}t}$$

where $i(0)$ is the initial current.

Consider the signal

$$x(t)=Ce^{at} \tag{I.23}$$

Let's assume that C and a are generally complex. Complex signals cannot appear in physical systems. However, the solutions of many differential equations are simplified by assuming that complex signals can appear both as excitations and in the solutions. Then, in translating the results back to physical systems, only the real part or the imaginary part of the solution is used.

An important relation that is often applied in analyses which involve complex exponential functions is Euler's relation, given by

$$e^{j\theta}=\cos\theta+j\sin\theta \tag{I.24}$$

$$e^{-j\theta}=\cos\theta-j\sin\theta \tag{I.25}$$

Adding (I.24) and (I.25), and dividing by 2, we get

$$\cos\theta=\frac{e^{j\theta}+e^{-j\theta}}{2} \tag{I.26}$$

Subtracting (I.25) from (I.24), and dividing by 2, we get

$$\sin\theta=\frac{e^{j\theta}-e^{-j\theta}}{j2} \tag{I.27}$$

The last four relations are so useful in signal and system analysis that they should be memorized.

A complex quantity A can be represented in terms of its real and imaginary parts as follows

$$A = A_R + jA_I \tag{I.28}$$

where A_R and A_I are, respectively, the real and imaginary parts of A, and are given by

$$A_R = \operatorname{Re}\{A\} \tag{I.29}$$

$$A_I = \operatorname{Im}\{A\} \tag{I.30}$$

Note that both A_R and A_I are real quantities.

A complex quantity A can also be represented in polar form as follows

$$A = |A|\angle\phi_A \tag{I.31}$$

where $|A|$ and ϕ_A are, respectively, the magnitude and phase of A, and are given by

$$|A| = \sqrt{A_R^2 + A_I^2} \tag{I.32}$$

$$\phi_A = \tan^{-1}\left(\frac{A_I}{A_R}\right) \tag{I.33}$$

If we apply the above to the complex exponential function in (I.24), we have

$$\begin{aligned} A &= e^{j\theta} \\ &= \cos\theta + j\sin\theta \end{aligned} \tag{I.34}$$

Substituting (I.34) into (I.32) and (I.33), we get

$$\begin{aligned} |A| &= \sqrt{\cos^2\theta + \sin^2\theta} \\ &= 1 \end{aligned} \tag{I.35}$$

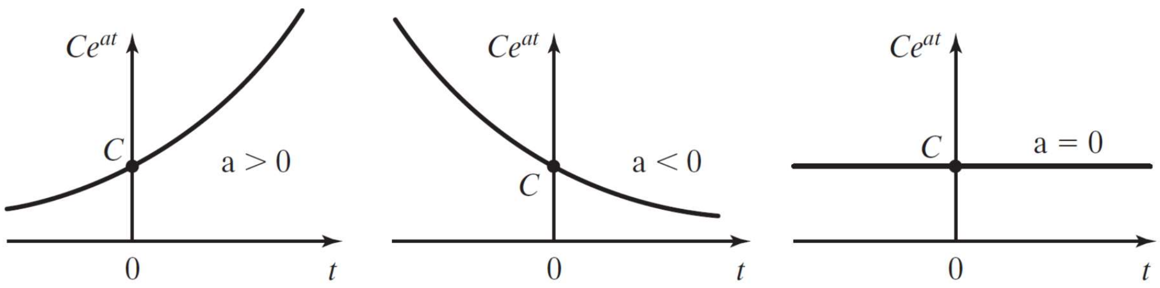
$$\begin{aligned} \phi_A &= \tan^{-1}\left(\frac{\sin\theta}{\cos\theta}\right) \\ &= \theta \end{aligned} \tag{I.36}$$

The complex exponential can then be expressed in polar form as

$$e^{j\theta} = 1\angle\theta \tag{I.37}$$

I.3.A. CASE 1: REAL C AND a

The signal $x(t) = Ce^{at}$ is plotted below for $C > 0$ with $a > 0$, $a < 0$ and $a = 0$.



For $a > 0$, the signal magnitude increases monotonically without limit with increasing time. For $a < 0$, the signal magnitude decreases monotonically toward zero as time increases. For $a = 0$, the signal is constant.

For $a < 0$, the signal decays toward zero, but does not reach zero in finite time. To differentiate between exponentials that decay at different rates, we let

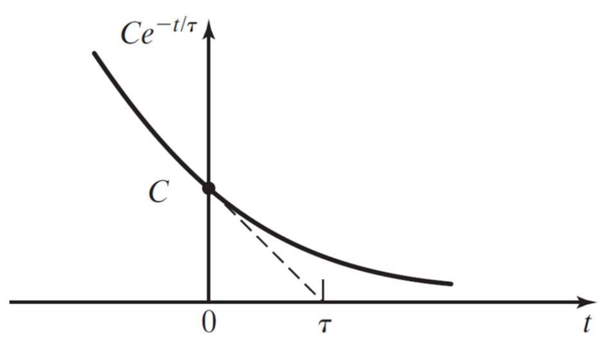
$$a = -\frac{1}{\tau} \tag{I.38}$$

where $\tau > 0$. Substituting (I.38) into (I.23), the signal $x(t)$ can be written in the form

$$\begin{aligned} x(t) &= Ce^{at} \\ &= Ce^{-\frac{t}{\tau}} \end{aligned} \tag{I.39}$$

The constant parameter τ is called the time constant of the exponential.

The time constant of an exponential signal is illustrated in the below figure.



Example

The signal $x(t) = 3e^{-4t}$ has a time constant $\tau = 0.25$ s.

Exercise

What is the value of the signal $x(t) = x(0)e^{-\frac{t}{\tau}}$ when $t = \tau$? What is the value of $\frac{x(\tau)}{x(0)}$?

The time derivative of $x(t)$ in (I.39) is given by

$$\frac{d}{dt}x(t) = -\frac{C}{\tau}e^{-\frac{t}{\tau}} \tag{I.40}$$

Evaluating the derivative at $t = 0$ yields

$$\left.\frac{d}{dt}x(t)\right|_{t=0} = -\frac{C}{\tau} \tag{I.41}$$

Equation (I.41) gives the rate of change of $x(t)$ at $t = 0$. Note that, according to (I.40), the rate of change is a function of time. If the rate of change were constant and equal to the one in (I.41), the signal would reach the zero value to $t = \tau$. In fact, the value of the signal at $t = \tau$ is equal to

$$\begin{aligned} x(\tau) &= Ce^{-1} \\ &= \frac{C}{e} \\ &\approx 0.368C \end{aligned} \tag{I.42}$$

Since $x(0) = C$,

$$\begin{aligned} x(\tau) &\approx 0.368C \\ &= 0.368x(0) \end{aligned} \tag{I.43}$$

In conclusion, the exponential signal decays to approximately 36.8% of its initial value after a time interval that is equal to the time constant.

Exercise

Determine $\frac{x(t_1 + \tau)}{x(t_1)}$ for any $t_1 > 0$.

The below table illustrates the decay of an exponential signal at integer multiples of the time constant. As can be seen in the table, the signal decays to less than 1% of its initial value in five time constants. In practice, the exponential signal can be assumed to have vanished after five time constants.

t	$e^{-t/\tau}$
0	1.0
τ	0.3679
2τ	0.1353
3τ	0.0498
4τ	0.0183
5τ	0.0067

I.3.B. Case 2: Complex C and Imaginary a

$$\begin{aligned} x(t) &= Ce^{at} \\ C &= A_C e^{j\phi_C} = A_C \angle \phi_C \\ a &= j\omega_0 \end{aligned} \tag{I.44}$$

where A_C, ϕ_C, and ω_0 are real and constant. The complex exponential signal $x(t)$ can be expressed as

$$\begin{aligned} x(t) &= A_C e^{j\phi_C} e^{j\omega_0 t} \\ &= A_C e^{j(\omega_0 t + \phi_C)} \\ &= A_C \cos(\omega_0 t + \phi_C) + jA_C \sin(\omega_0 t + \phi_C) \end{aligned} \tag{I.45}$$

The signal $x(t)$ is periodic. Its fundamental angular frequency is ω_0. The fundamental frequency is $f_0 = \omega_0 / 2\pi$. The fundamental period is $T_0 = 1/f_0 = 2\pi/\omega_0$.

The real part of $x(t)$ is given by

$$\begin{aligned} x_R(t) &= \text{Re}\{x(t)\} \\ &= A_C \cos(\omega_0 t + \phi_C) \end{aligned} \tag{I.46}$$

The signal $x_R(t)$ is plotted in Figure I.7.

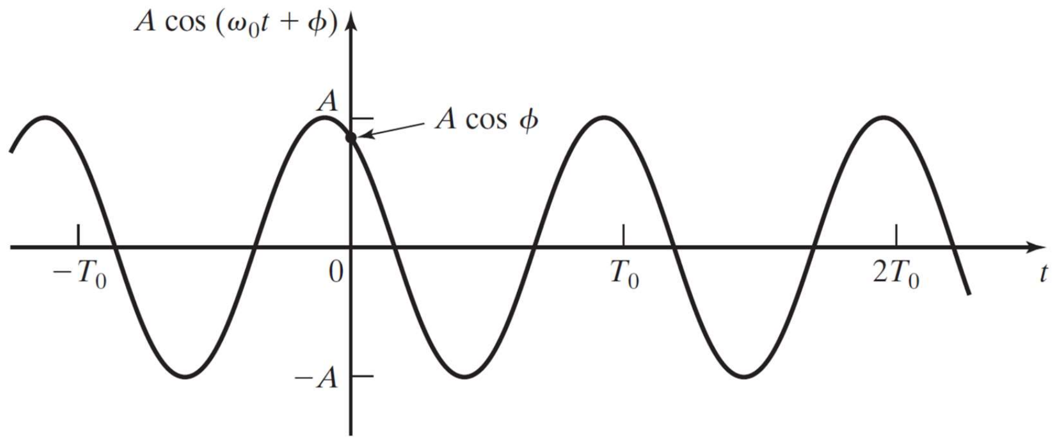


Figure I.7: Real part of complex exponential signal

Harmonically Related Complex Exponentials

Harmonically related complex exponentials are a set of functions with frequencies related by integers, of the form

$$x_k(t) = A_k e^{jk\omega_0 t}, \quad k = \pm 1, \pm 2, \ldots. \tag{2.28}$$

We will make extensive use of harmonically related complex exponentials later when we study the Fourier series representation of periodic signals.

CASE 3

Both *C* and *a* Complex

For this case, the complex exponential $x(t) = Ce^{at}$ has the parameters

$$x(t) = Ce^{at}; \quad C = Ae^{j\phi}; \quad a = \sigma_0 + j\omega_0, \tag{2.29}$$

where A, ϕ, σ, and ω_0 are real and constant. The complex exponential signal can then be expressed as

$$\begin{aligned} x(t) &= Ae^{j\phi} e^{(\sigma_0 + j\omega_0)t} = Ae^{\sigma_0 t} e^{j(\omega_0 t + \phi)} \\ &= Ae^{\sigma_0 t}\cos(\omega_0 t + \phi) + jAe^{\sigma_0 t}\sin(\omega_0 t + \phi) \\ &= x_r(t) + jx_i(t). \end{aligned} \tag{2.30}$$

In this expression, both $x_r(t) = \text{Re}[x(t)]$ and $x_i(t) = \text{Im}[x(t)]$ are real. The notation Re[•] denotes the real part of the expression, and Im[•] denotes the imaginary part. Plots of the real part of (2.30) are given in Figure 2.16 for $\phi = 0$. Figure 2.16(a) shows the case that $\sigma_0 > 0$. Figure 2.16(b) shows the case that $\sigma_0 < 0$; this signal is called an *underdamped sinusoid*. For $\sigma_0 = 0$ as in Figure 2.15, the signal is called an *undamped sinusoid*.

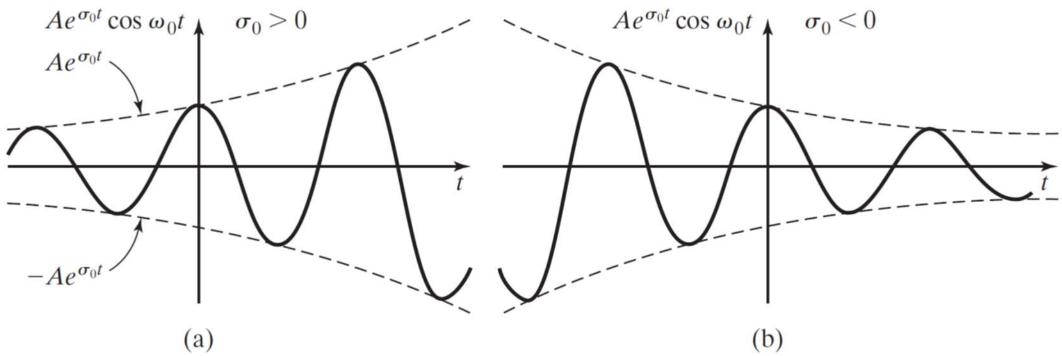


Figure 2.16 Real part of a complex exponential.

In Figure 2.16(a), by definition, the *envelope* of the signal is $\pm Ae^{\sigma_0 t}$. Because both the cosine function and the sine function have magnitudes that are less than or equal to unity, in (2.30),

$$-Ae^{\sigma_0 t} \leqq x_r(t) \leqq Ae^{\sigma_0 t}, \quad -Ae^{\sigma_0 t} \leqq x_i(t) \leqq Ae^{\sigma_0 t}. \tag{2.31}$$

For the case that the sinusoid is damped ($\sigma_0 < 0$), the envelope can be expressed as $\pm Ae^{-t/\tau}$; we say that this damped sinusoid has a time constant of τ seconds.

SINGULARITY FUNCTIONS

In this section, we consider a class of functions called singularity functions. We define a *singularity function* as one that is related to the impulse function (to be defined in this section) and associated functions. Two singularity functions are emphasized in this section: the unit step function and the unit impulse function. We begin with the unit step function.

Unit Step Function

The unit step function, denoted as $u(t)$, is usually employed to switch other signals on or off. The *unit step function* is defined as

$$u(\tau) = \begin{cases} 1, & \tau > 0 \\ 0, & \tau < 0 \end{cases}, \tag{2.32}$$

where the independent variable is denoted as τ. In the study of signals, we choose the independent variable to be a linear function of time. For example, if $\tau = (t - 5)$, the unit step is expressed as

$$u(t - 5) = \begin{cases} 1, & t - 5 > 0 \Rightarrow t > 5 \\ 0, & t - 5 < 0 \Rightarrow t < 5 \end{cases}.$$

This unit step function has a value of unity for $t > 5$ and a value of zero for $t < 5$. The general unit step is written as $u(t - t_0)$, with

$$u(t - t_0) = \begin{cases} 1, & t > t_0 \\ 0, & t < t_0 \end{cases}.$$

A plot of $u(t - t_0)$ is given in Figure 2.17 for a value of $t_0 > 0$.

The unit step function has the property

$$u(t - t_0) = [u(t - t_0)]^2 = [u(t - t_0)]^k, \tag{2.33}$$

with k any positive integer. This property is based on the relations $(0)^k = 0$ and $(1)^k = 1, k = 1, 2, \ldots$. A second property is related to time scaling:

$$u(at - t_0) = u(t - t_0/a), a \neq 0. \tag{2.34}$$

(See Problem 2.21.)

Note that we have not defined the value of the unit step function at the point that the step occurs. Unfortunately, no standard definition exists for this value. As is sometimes done, we leave this value undefined; some authors define the value as zero, and some define it as one-half, while others define the value as unity.

u(t+T/2) → t0= - T/2

-u(t-T/2) → t0= T/2

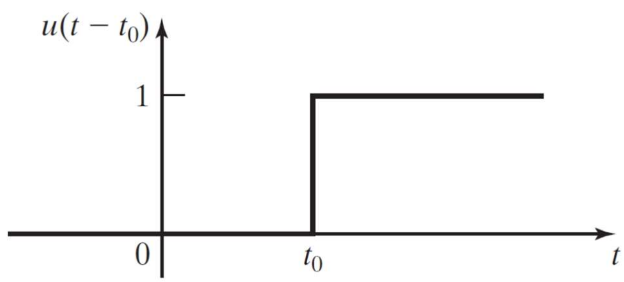


Figure 2.17 Unit step function.

As previously stated, the unit step is often used to switch functions. An example is given by

$$\cos \omega t\, u(t) = \begin{cases} \cos \omega t, & t > 0 \\ 0, & t < 0 \end{cases}.$$

The unit step allows us mathematically to switch this sinusoidal function on at $t = 0$. Another example is $v(t) = 12u(t)$ volts; this function is equal to 0 volts for $t < 0$ and to 12 volts for $t > 0$. In this case, the unit step function is used to switch a 12-V source.

Another useful switching function is the unit rectangular pulse, rect(t/T), which is *defined* as

$$\text{rect}(t/T) = \begin{cases} 1, & -T/2 < t < T/2 \\ 0, & \text{otherwise} \end{cases}.$$

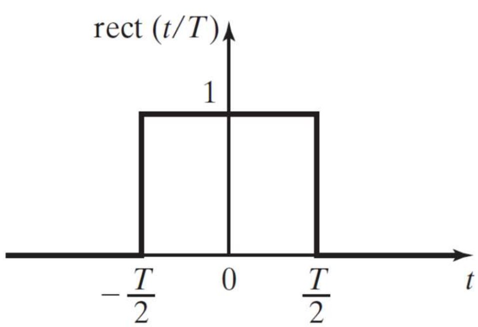
This function is plotted in Figure 2.18(a). It can be expressed as three different functions of unit step signals:

$$\text{rect}(t/T) = \begin{cases} u(t + T/2) - u(t - T/2) \\ u(T/2 - t) - u(-T/2 - t). \\ u(t + T/2)u(T/2 - t) \end{cases} \quad (2.35)$$

These functions are plotted in Figure 2.18(b), (c), and (d).

rect((t-T/2)/T): from 0 to T.

rect((t+T/2)/T): from -T to 0.



(a)

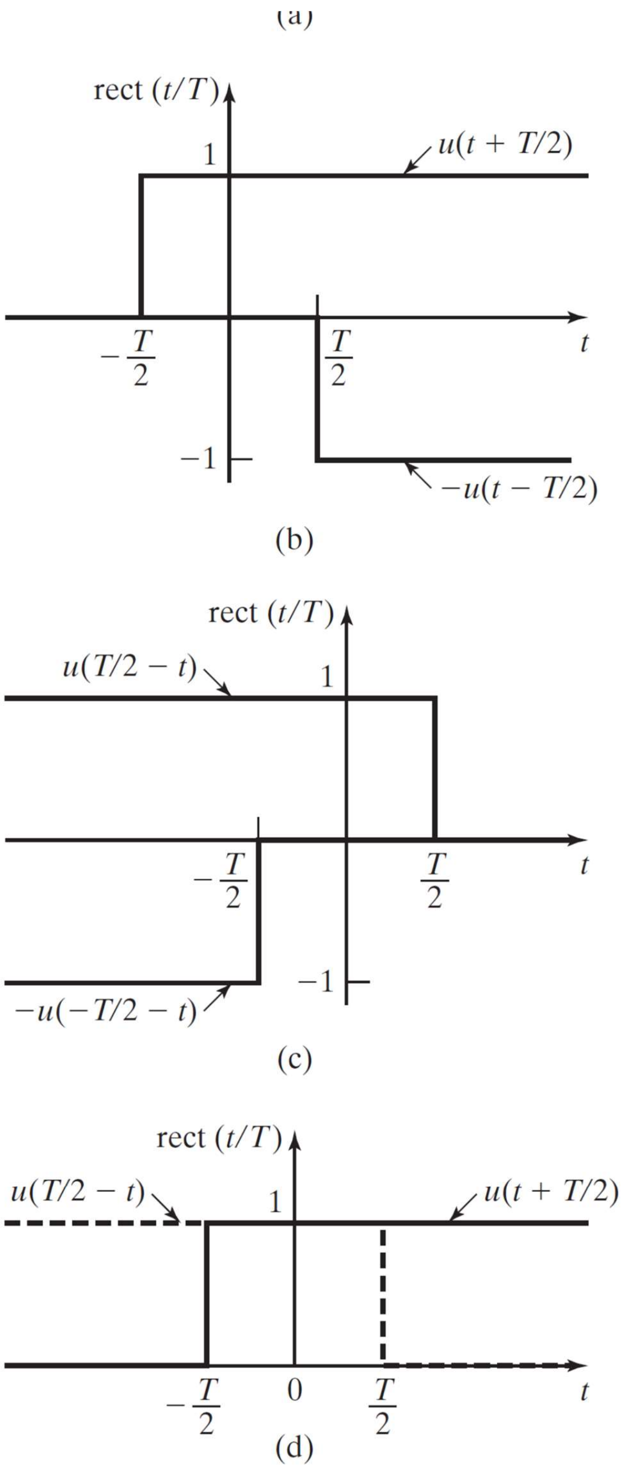


Figure 2.18 Unit rectangular pulse.

The time-shifted rectangular pulse function is given by

$$\text{rect}[(t - t_0)/T] = \begin{cases} 1, & t_0 - T/2 < t < t_0 + T/2 \\ 0, & \text{otherwise} \end{cases}. \tag{2.36}$$

This function is plotted in Figure 2.19. Notice that in both (2.35) and (2.36) the rectangular pulse has a duration of T seconds.

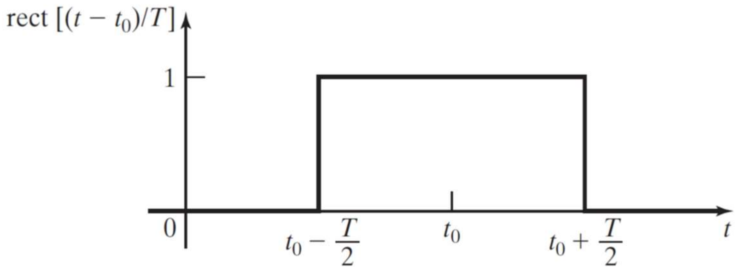


Figure 2.19 Time-shifted rectangular function.

The unit rectangular pulse is useful in extracting part of a signal. For example, the signal $x(t) = \cos t$ has a period $T_0 = 2\pi/\omega = 2\pi$. Consider a signal composed of one period of this cosine function beginning at $t = 0$, and zero for all other time. This signal can be expressed as

$$x(t) = (\cos t)[u(t) - u(t - 2\pi)] = \begin{cases} \cos t, & 0 < t < 2\pi \\ 0, & \text{otherwise} \end{cases}.$$

The rectangular-pulse notation allows us to write

$$x(t) = \cos t \ \text{rect}[(t - \pi)/2\pi].$$

This sinusoidal pulse is plotted in Figure 2.20. Another example of writing the equation of a signal using unit step functions will now be given.

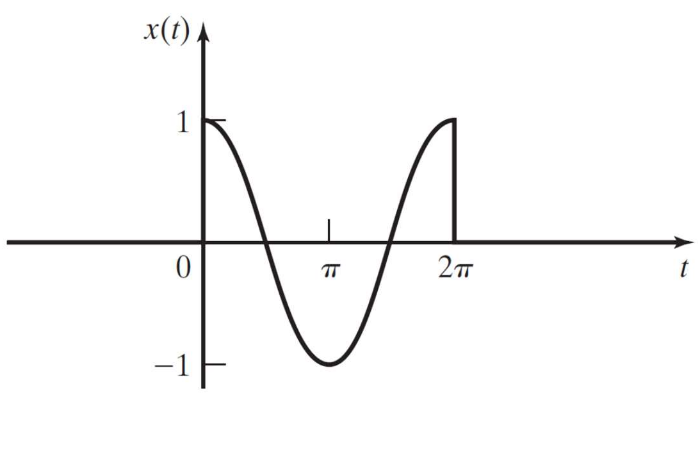


Figure 2.20 The function $x(t) = \cos t \ \text{rect}[(t - \pi)/2\pi]$.

EXAMPLE 2.9 **Equations for a half-wave rectified signal**

Consider again the half-wave rectified signal described in Section 1.2 and Example 2.6, and shown again as $v(t)$ in Figure 2.21. We assume that $v(t)$ is zero for $t < 0$ in this example. If a system containing this signal is to be analyzed or designed, the signal must be expressed as a mathematical function. In Figure 2.21, the signal for $0 \leqq t \leqq T_0$ can be written as

$$\begin{aligned} v_1(t) &= (V_m \sin \omega_0 t)[u(t) - u(t - T_0/2)] \\ &= V_m \sin(\omega_0 t)\, \mathrm{rect}[(t - T_0/4)/(T_0/2)], \end{aligned}$$

where $T_0 = 2\pi/\omega_0$. This signal, $v_1(t)$, is equal to the half-wave rectified signal for $0 \leqq t \leqq T_0$ and is zero elsewhere. Thus, the half-wave rectified signal can be expressed as a sum of shifted signals,

$$\begin{aligned} v(t) &= v_1(t) + v_1(t - T_0) + v_1(t - 2T_0) + \cdots \\ &= \sum_{k=0}^{\infty} v_1(t - kT_0), \end{aligned} \tag{2.37}$$

since $v_1(t - T_0)$ is $v_1(t)$ delayed by one period, $v_1(t - 2T_0)$ is $v_1(t)$ delayed by two periods, and so on. If the half-wave rectified signal is specified as periodic for all time, the lower limit in (2.37) is changed to negative infinity. As indicated in this example, expressing a periodic signal as a mathematical function often requires the summation of an infinity of terms. ■

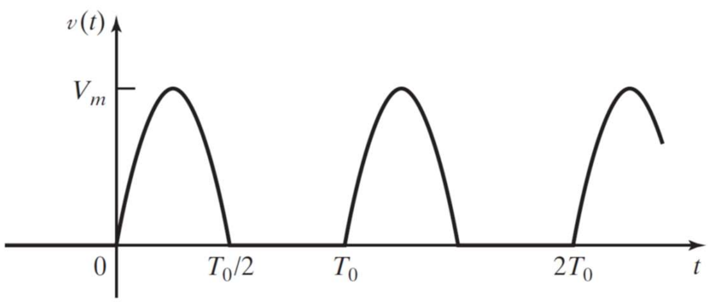


Figure 2.21 Half-wave rectified signal.

Unit Impulse Function

Engineers have found great use for $j = \sqrt{-1}$, even though this is not a real number and cannot appear in nature. Electrical engineering analysis and design utilizes j extensively. In the same manner, engineers have found great use for the *unit impulse function*, $\delta(t)$, even though this function cannot appear in nature. In fact, the impulse function is not a mathematical function in the usual sense [2]. The unit impulse function is also called the *Dirac delta function.* The impulse function was introduced by Nobel Prize winning physicist Paul Dirac.

To introduce the impulse function, we begin with the integral of the unit step function; this integral yields the unit ramp function

$$f(t) = \int_0^t u(\tau - t_0)d\tau = \int_{t_0}^t d\tau = \tau\Big|_{t_0}^t = [t - t_0]u(t - t_0), \tag{2.38}$$

where $[t - t_0]u(t - t_0)$, by definition, is the *unit ramp function*. In (2.38), the factor $u(t - t_0)$ in the result is necessary, since the value of the integral is zero for $t < t_0$. The unit step function and the unit ramp function are illustrated in Figure 2.22.

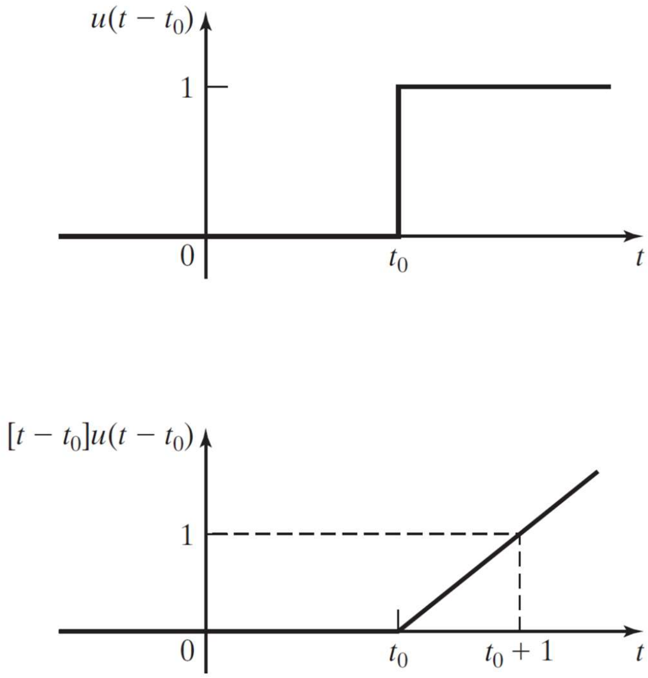


Figure 2.22 Integral of the unit step function.

Note that in Figure 2.22 and in (2.38), the unit step function is the derivative of the unit ramp function. We have no mathematical problems in (2.38) or in the derivative of (2.38). However, problems do occur if we attempt to take the second derivative of (2.38). We now consider this derivative.

The result of differentiating the unit step function $u(t - t_0)$ is not a function in the usual mathematical sense. The derivative is undefined at the only point, $t = t_0$, where it is not zero. (See Figure 2.22.) However, this derivative has been shown, by the rigorous mathematical theory of distributions [2–6], to be very useful in the modeling and analysis of systems. We now consider this derivative as the limit of a derivative that does exist.

No signal can change instantaneously in a physical system, since this change, in general, represents an instantaneous transfer of energy. Hence, we can consider the function $f(t)$ in Figure 2.23(a) to be a more accurate model of a physical step function. We can differentiate this function and the resulting derivative is the rectangular pulse $g(t)$ of Figure 2.23(b); that is,

$$g(t) = \frac{df(t)}{dt}.$$

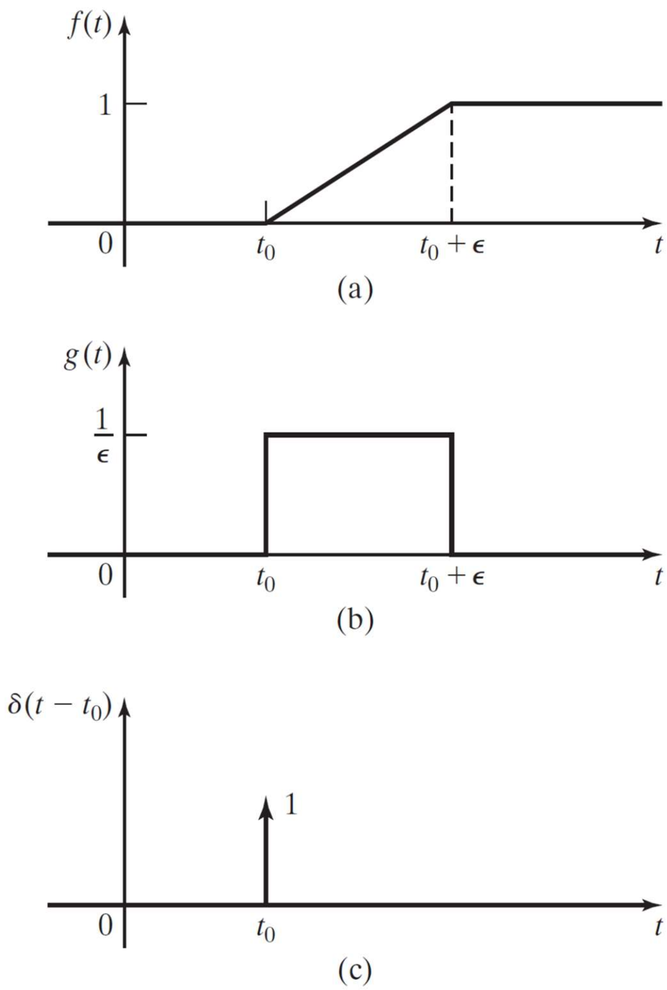


Figure 2.23 Generation of an impulse function.

The practical function $f(t)$ in Figure 2.23(a) approaches the unit step function $u(t - t_0)$ if we allow ϵ to approach zero. For this case, the width of $g(t)$ approaches zero and the amplitude becomes unbounded. However, the area under $g(t)$ remains constant at unity, since this area is independent of the pulse width ϵ.

We often call the limit of $g(t)$ in Figure 2.23(b) as ϵ approaches zero the *unit impulse function*. Hence, with $\delta(t - t_0)$ denoting the unit impulse function, we can employ the concept that

$$\lim_{\epsilon \to 0} g(t) = \delta(t - t_0) \tag{2.39}$$

to convey a mental image of the unit impulse function. However, the impulse function is not a function in the ordinary sense, since it is zero at every point except t_0, where it is unbounded. However, the area under a unit impulse function is well defined and is equal to unity. On the basis of these properties, we *define* the unit impulse function $\delta(t - t_0)$ by the relations

$$\delta(t - t_0) = 0, \quad t \neq t_0;$$
$$\int_{-\infty}^{\infty} \delta(t - t_0)dt = 1. \tag{2.40}$$

We depict the impulse function as a vertical arrow as shown in Figure 2.23(c), where the number written beside the arrow denotes the multiplying constant of the unit impulse function. For example, for the function $5\delta(t - t_0)$, that number is 5. This multiplying constant is called the *weight* of the impulse. The *amplitude* of the impulse function at $t = t_0$ is unbounded, while the multiplying factor (the weight) is the *area* under the impulse function.

We say that the impulse function $\delta(t - t_0)$ "occurs" at $t = t_0$ because this concept is useful. The quotation marks are used because the impulse function (1) is not an ordinary function and (2) is defined rigorously only under the integral in (2.41). The operation in (2.41) is often taken one step further; if $f(t)$ is continuous at $t = t_0$, then

$$f(t)\delta(t - t_0) = f(t_0)\delta(t - t_0). \tag{2.42}$$

The definition of the impulse function (2.40) is not mathematically rigorous [3]; we now give the definition that is. For any function $f(t)$ that is continuous at $t = t_0$, $\delta(t - t_0)$ is *defined* by the integral

$$\int_{-\infty}^{\infty} f(t)\delta(t - t_0)dt = f(t_0). \tag{2.41}$$

The product of a continuous-time function $f(t)$ and $\delta(t - t_0)$ is an impulse with its weight equal to $f(t)$ evaluated at time t_0, the time that the impulse occurs. Equations (2.41) and (2.42) are sometimes called the *sifting* property of the impulse function. This result can be reasoned by considering the impulse function, $\delta(t - t_0)$, to have a value of zero, except at $t = t_0$. Therefore, the only value of $f(t)$ that is significant in the product $f(t)\delta(t - t_0)$ is the value at $t = t_0$, $f(t_0)$.

Table 2.3 lists the definition and several properties of the unit impulse function. See Refs. 2 through 6 for rigorous proofs of these properties. The properties listed in Table 2.3 are very useful in the signal and system analysis to be covered later.

TABLE 2.3 Properties of the Unit Impulse Function

1.	$\int_{-\infty}^{\infty} f(t)\delta(t - t_0)dt = f(t_0)$, $f(t)$ continuous at $t = t_0$
2.	$\int_{-\infty}^{\infty} f(t - t_0)\delta(t)dt = f(-t_0)$, $f(t)$ continuous at $t = -t_0$
3.	$f(t)\delta(t - t_0) = f(t_0)\delta(t - t_0)$, $f(t)$ continuous at $t = t_0$
4.	$\delta(t - t_0) = \frac{d}{dt}u(t - t_0)$
5.	$u(t - t_0) = \int_{-\infty}^{t} \delta(\tau - t_0)d\tau = \begin{cases} 1, & t > t_0 \\ 0, & t < t_0 \end{cases}$
6.	$\int_{-\infty}^{\infty} \delta(at - t_0)dt = \frac{1}{\lvert a \rvert}\int_{-\infty}^{\infty} \delta\left(t - \frac{t_0}{a}\right)dt$
7.	$\delta(-t) = \delta(t)$

EXAMPLE 2.10 **Integral evaluations for impulse functions**

This example illustrates the evaluation of some integrals containing impulse functions, using Table 2.3, for $f(t)$ given in Figure 2.24(a). First, from Property 1 in Table 2.3 with $t_0 = 0$,

$$\int_{-\infty}^{\infty} f(t)\delta(t)dt = f(0) = 2,$$

and the value of the integral is equal to the value of $f(t)$ at the point at which the impulse function occurs. Next, for Figure 2.24(b), from Property 2 in Table 2.3,

$$\int_{-\infty}^{\infty} f(t - 1)\delta(t)dt = f(-1) = 3.$$

As a third example, for Figure 2.24(c),

$$\int_{-\infty}^{\infty} f(t)\delta(t-1)dt = f(1) = 1,$$

from Property 1 in Table 2.3. For Figure 2.24(d),

$$\int_{-\infty}^{\infty} f(t-1)\delta(t-1)dt = f(0) = 2,$$

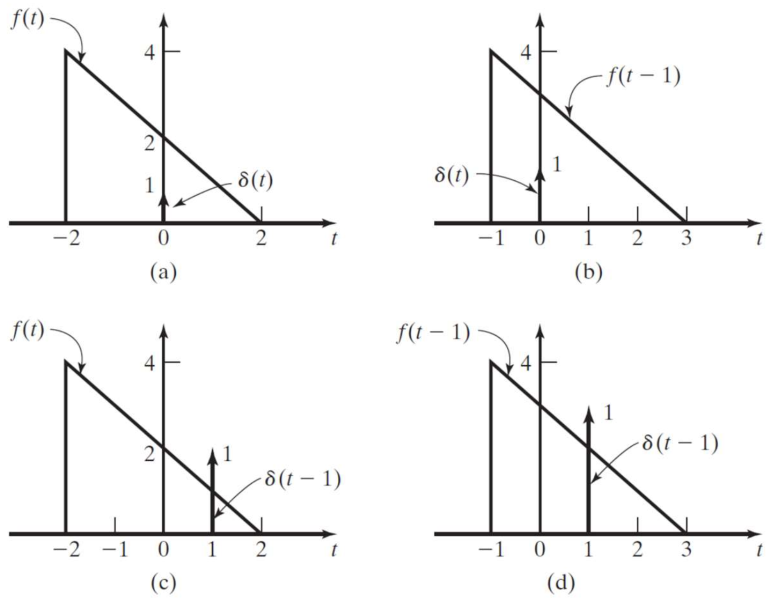


Figure 2.24 Signals for Example 2.10.

from Property 1 in Table 2.3. We have considered all possible combinations of delaying the functions. In each case, the value of the integral is that value of $f(t)$ at which the impulse function occurs.

As a final example, consider the effects of time scaling the impulse function,

$$\int_{-\infty}^{\infty} f(t)\delta(4t)dt = \frac{1}{4}\int_{-\infty}^{\infty} f(t)\delta(t)dt = \frac{f(0)}{4} = \frac{1}{2},$$

from Property 6 in Table 2.3. ■

2.5 MATHEMATICAL FUNCTIONS FOR SIGNALS

In Example 2.9, we wrote the equation of a half-wave rectified signal by using unit step functions. This topic is considered further in this section. First, we consider an example.

EXAMPLE 2.11 **Plotting signal waveforms**

In this example, we consider a signal given in mathematical form:

$$f(t) = 3u(t) + tu(t) - [t - 1]u(t - 1) - 5u(t - 2). \qquad (2.43)$$

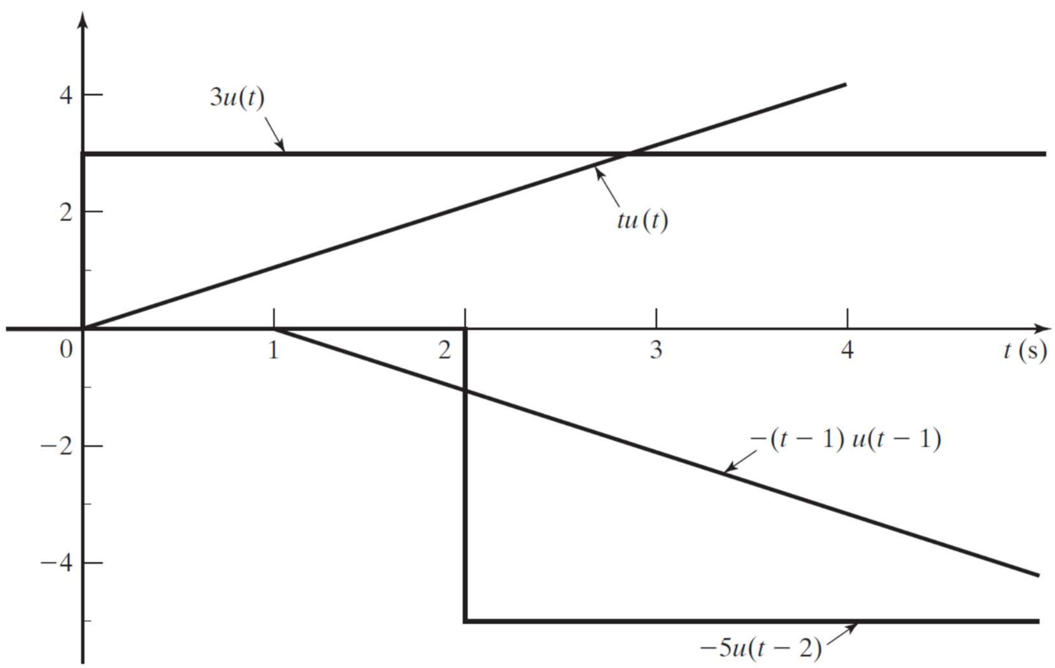
The terms of $f(t)$ are plotted in Figure 2.25(a), and $f(t)$ is plotted in Figure 2.25(b). We now verify these plots. The four terms of $f(t)$ are evaluated as

$$3u(t) = \begin{cases} 3, & t > 0 \\ 0, & t < 0 \end{cases};$$

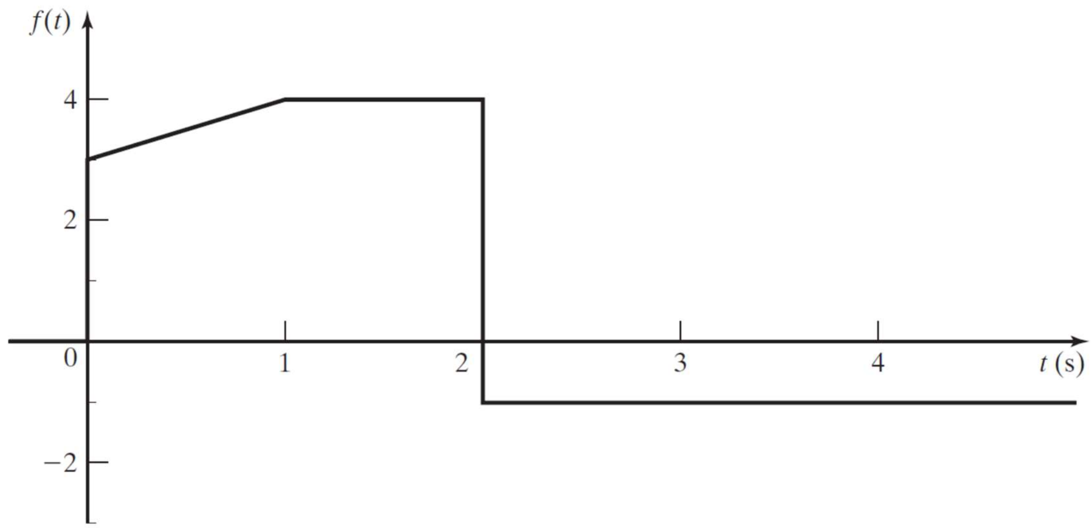
$$tu(t) = \begin{cases} t, & t > 0 \\ 0, & t < 0 \end{cases};$$

$$(t - 1)u(t - 1) = \begin{cases} t - 1, & t > 1 \\ 0, & t < 1 \end{cases};$$

$$5u(t - 2) = \begin{cases} 5, & t > 2 \\ 0, & t < 2 \end{cases}.$$



(a)



(b)

Figure 2.25 Signal for Example 2.11.

Using these functions and (2.43), we can write the equations for $f(t)$ (as the sum of four terms) over each different range:

$$\begin{aligned} t < 0, &\quad f(t) = 0 + 0 - 0 - 0 = 0\,; \\ 0 < t < 1, &\quad f(t) = 3 + t - 0 - 0 = 3 + t\,; \\ 0 < t < 2, &\quad f(t) = 3 + t - (t-1) - 0 = 4\,; \\ 2 < t, &\quad f(t) = 3 + t - (t-1) - 5 = -1\,. \end{aligned}$$

The graph of $f(t)$ given in Figure 2.25(b) is correct. ■

EXAMPLE 2.12 **Equations for straight-line-segments signal**

The equation for the signal in Figure 2.27 will be written. The slope of the signal changes from 0 to 3 for a change in slope of 3, beginning at $t = -2$. The slope changes from 3 to -3 at $t = -1$, for a change in slope of -6. At $t = 1$, the slope becomes 0 for a change in slope of 3.

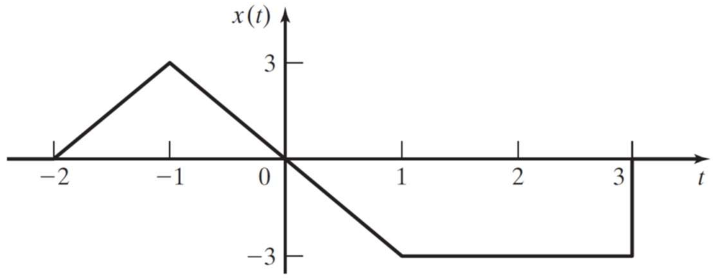


Figure 2.27 Signal for Example 2.12.

The function steps from -3 to 0 at $t = 3$, for a change in amplitude of 3. Hence, the equation for $x(t)$ is given by

$$x(t) = 3[t + 2]u(t + 2) - 6[t + 1]u(t + 1) + 3[t - 1]u(t - 1) + 3u(t - 3).$$

***

II. Continuous Time Systems

System
A *system* is a process for which cause-and-effect relations exist.

For our purposes, the cause is the system input signal, the effect is the system output signal, and the relations are expressed as equations (the system model). We often refer to the input signal and the output signal as simply the input and the output, respectively.

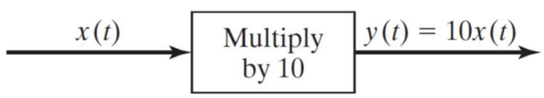


Figure 2.31 Ideal amplifier.

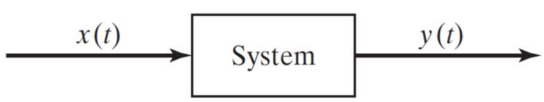


Figure 2.32 Representation of a general system.

One representation of a general system is by a block diagram as shown in Figure 2.32. The input signal is $x(t)$, and the output signal is $y(t)$. The system may be denoted by the equation

$$y(t) = T[x(t)], \tag{2.51}$$

where the notation T[·] indicates a *transformation*. This notation does not indicate a function; that is, T[$x(t)$] is not a mathematical function into which we substitute $x(t)$ and directly calculate $y(t)$. The explicit set of equations relating the input $x(t)$ and the output $y(t)$ is called the *mathematical model,* or simply, the model, of the system. Given the input $x(t)$, this set of equations must be solved to obtain $y(t)$. For continuous-time systems, the model is usually a set of *differential equations*.

EXAMPLE 2.15 **Transformation notation for a circuit**

Consider the circuit of Figure 2.33. We define the system input as the voltage source $v(t)$ and the system output as the inductor voltage $v_L(t)$. The *transformation notation* for the system is

$$v_L(t) = T[v(t)]. \tag{2.52}$$

The equations that model the system are given by

$$L\frac{di(t)}{dt} + Ri(t) = v(t)$$

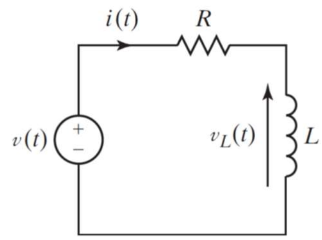


Figure 2.33 *RL* circuit.

$$v_L(t) = L\frac{di(t)}{dt}. \tag{2.53}$$

Hence, the transformation notation of (2.52) represents the explicit equations of (2.53). This model, (2.53), is two equations, with the first a first-order linear differential equation with constant coefficients. (L and R are constants.)

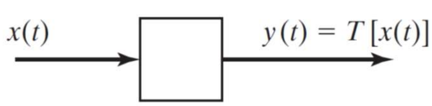
Interconnecting Systems

In this section, the system-transformation notation of (2.51) will be used to specify the interconnection of systems. First, we define three block-diagram elements. The first element is a block as shown in Figure 2.34(a); this block is a graphical representation of a system described by (2.51). The second element is a circle that represents a summing junction as shown in Figure 2.34(b). The output signal of the junction is defined to be the sum of the input signals. The third element is a circle that represents a product junction, as shown in Figure 2.34(c). The output signal of the junction is defined to be the product of the input signals.

We next define two basic connections for systems. The first is the *parallel* connection and is illustrated in Figure 2.35(a). Let the output of System 1 be $y_1(t)$ and that of System 2 be $y_2(t)$. The output signal of the total system is then given by

$$y(t) = y_1(t) + y_2(t) = T_1[x(t)] + T_2[x(t)] = T[x(t)]. \tag{2.54}$$

The notation for the total system is $y(t) = T[x(t)]$.



(a)

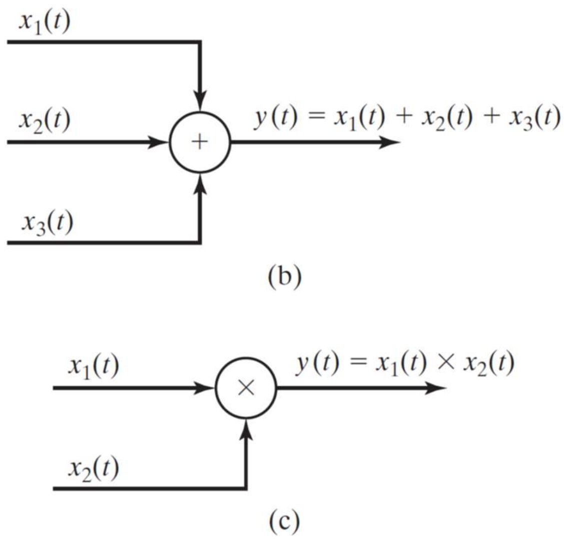


Figure 2.34 Block-diagram elements.

The second basic connection for systems is illustrated in Figure 2.35(b). This connection is called the *series,* or *cascade, connection*. In this figure, the output signal of the first system is $y_1(t) = T_1[x(t)]$, and the total-system output signal is

$$y(t) = T_2[y_1(t)] = T_2(T_1[x(t)]) = T[x(t)]. \qquad (2.55)$$

The system equations of (2.54) and (2.55) cannot be simplified further until the mathematical models for the two systems are known.

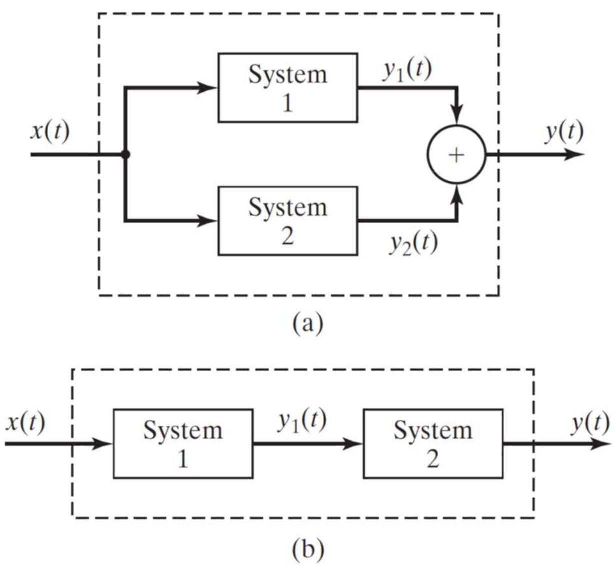


Figure 2.35 Basic connections of systems.

EXAMPLE 2.16 **Interconnections for a system**

Consider the system of Figure 2.37. Each block represents a system, with a number given to identify the system. The circle with the symbol × denotes the multiplication of the two input signals. We can write the following equations for the system:

$$y_3(t) = y_1(t) + y_2(t) = T_1[x(t)] + T_2[x(t)]$$

and

$$y_4(t) = T_3[y_3(t)] = T_3(T_1[x(t)] + T_2[x(t)]).$$

Thus,

$$y(t) = y_4(t) \times y_5(t) = [T_3(T_1[x(t)] + T_2[x(t)])]T_4[x(t)]. \quad (2.56)$$

This equation denotes only the interconnection of the systems. The mathematical model of the total system depends on the mathematical models of the individual systems.

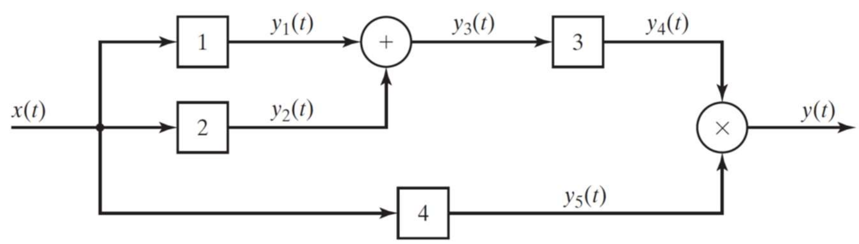


Figure 2.37 System for Example 2.16.

The basic configuration of a feedback-control system is given in Figure 2.38. The *plant* is the physical system to be controlled. The *controller* is a physical system inserted by the design engineers to give the total system certain desired characteristics. The *sensor* measures the signal to be controlled, and the input signal represents the desired output. The error signal $e(t)$ is a measure of the difference between the desired output, modeled as $x(t)$, and the measurement of the output $y(t)$. We write the equations of this system as

$$e(t) = x(t) - T_3[y(t)]$$

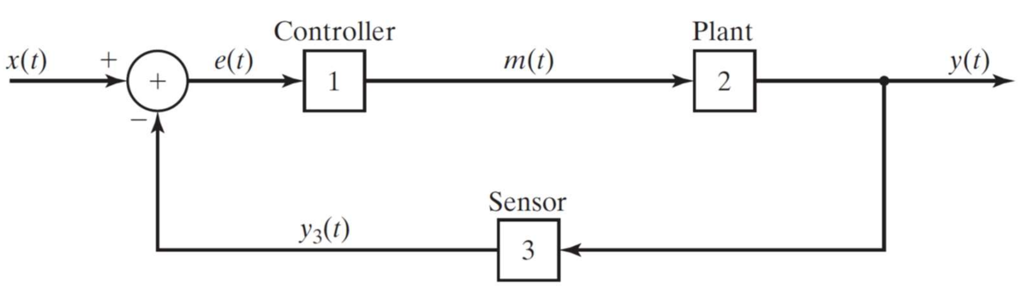


Figure 2.38 Feedback-control system.

and

$$y(t) = T_2[m(t)] = T_2(T_1[e(t)]). \quad (2.57)$$

Hence, we can express the output signal as

$$y(t) = T_2[T_1(x(t) - T_3[y(t)])]. \quad (2.58)$$

The system output is expressed as a function of both the system input and the system output, as is always the case for feedback systems. We cannot further simplify relationship (2.58) without knowing the mathematical models of the three subsystems. A simple example of the model of a feedback control system is now given.

EXAMPLE 2.17 **Interconnections for a feedback system**

In the feedback-control system of Figure 2.38, suppose that the controller and the sensor can be modeled as simple gains (amplitude scaling). These models are adequate in some physical control systems. Thus,

$$m(t) = T_1[e(t)] = K_1 e(t)$$

and

$$y_3(t) = T_3[y(t)] = K_3 y(t),$$

where K_1 and K_3 are real constants. Now,

$$e(t) = x(t) - K_3 y(t),$$

and thus,

$$m(t) = K_1 e(t) = K_1 x(t) - K_1 K_3 y(t).$$

Suppose also that the plant is modeled as a first-order differentiator such that

$$y(t) = T_2[m(t)] = \frac{dm(t)}{dt}.$$

Hence,

$$y(t) = \frac{d}{dt}[K_1 x(t) - K_1 K_3 y(t)] = K_1 \frac{dx(t)}{dt} - K_1 K_3 \frac{dy(t)}{dt}.$$

This equation is the system model and can be expressed as

$$K_1K_3\frac{dy(t)}{dt} + y(t) = K_1\frac{dx(t)}{dt}.$$

The system is modeled by a first-order linear differential equation with constant coefficients. ■

2.7 PROPERTIES OF CONTINUOUS-TIME SYSTEMS

In Section 2.6, continuous-time systems were introduced. In this section, we define some of the properties, or characteristics, of these systems. These definitions allow us to test the mathematical representation of a system to determine its properties.

When testing for the existence of a property, it is often much easier to establish that a system does not exhibit the property in question. To prove that a system does not have a particular property, we need to show only one counterexample. To prove that a system does have the property, we must present an analytical argument that is valid for an arbitrary input.

In the following relation, $x(t)$ denotes the input signal and $y(t)$ denotes the output signal of a system.

$$x(t) \to y(t). \tag{2.59}$$

We read this notation as "$x(t)$ produces $y(t)$"; it has the same meaning as the block diagram of Figure 2.32 and the transformation notation

[eq(2.51)] $$y(t) = T[x(t)].$$

The following are the six properties of continuous-time systems:

Memory
A system has memory if its output at time t_0, $y(t_0)$, depends on input values other than $x(t_0)$. Otherwise, the system is memoryless.

A system with memory is also called a *dynamic system*. An example of a system with memory is an *integrating amplifier,* described by

$$y(t) = K\int_{-\infty}^{t} x(\tau)d\tau. \tag{2.60}$$

$$v(t) = \frac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau.$$

A memoryless system is also called a *static system*. An example of a memoryless system is the ideal amplifier defined earlier. With $x(t)$ as its input and $y(t)$ as its output, the model of an ideal amplifier with (constant) gain K is given by

$$y(t) = Kx(t)$$

for all t. A second example is resistance, for which $v(t) = Ri(t)$. A third example is a squaring circuit, such that

$$y(t) = x^2(t). \tag{2.61}$$

Clearly, a system $y_1(t) = 5x(t)$ would be memoryless, whereas a second system $y_2(t) = x(t + 5)$ has memory, because $y_2(t_0)$ depends on the value of $x(t_0 + 5)$, which is five units of time ahead of t_0.

Invertibility
A system is said to be invertible if distinct inputs result in distinct outputs.

For an invertible system, the system input can be determined uniquely from its output. As an example, consider the squaring circuit mentioned earlier, which is described by

$$y(t) = x^2(t) \Rightarrow x(t) = \pm\sqrt{y(t)}. \tag{2.62}$$

Suppose that the output of this circuit is constant at 4 V. The input could be either +2 V or −2 V. Hence, this system is not invertible. An example of an invertible system is an ideal amplifier of gain K:

$$y(t) = Kx(t) \Rightarrow x(t) = \frac{1}{K}y(t). \tag{2.63}$$

Inverse of a System
The inverse of a system (denoted by T) is a second system (denoted by T_i) that, when cascaded with the system T, yields the identity system.

The notation for an inverse transformation is then

$$y(t) = T[x(t)] \Rightarrow x(t) = T_i[y(t)]. \tag{2.64}$$

Hence, $T_i[\cdot]$ denotes the inverse transformation. If a system is invertible, we can find the unique $x(t)$ for each $y(t)$ in (2.64). We illustrate an invertible system in Figure 2.39. In this figure,

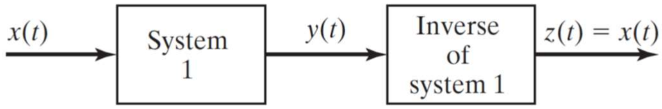


Figure 2.39 Inverse system.

$$z(t) = T_2[y(t)] = T_{i1}(T_1[x(t)]) = x(t), \tag{2.65}$$

where $T_2(\cdot) = T_{i1}(\cdot)$, the inverse of system $T_1(\cdot)$.

A simple example of the inverse of a system is an ideal amplifier with gain 5. Note that we can obtain the inverse system by solving for $x(t)$ in terms of $y(t)$:

$$y(t) = T[x(t)] = 5x(t) \Rightarrow x(t) = T_i[y(t)] = 0.2y(t). \tag{2.66}$$

The inverse system is an ideal amplifier with gain 0.2.

Causality

A system is causal if the output at any time t_0 is dependent on the input only for $t \leq t_0$.

A causal system is also called a *nonanticipatory system. All physical systems are causal*.

A *filter* is a physical device (system) for removing certain unwanted components from a signal. We can design better filters for a signal if all past values and all future values of the signal are available. In real time (as the signal occurs in the physical system), we never know the future values of a signal. However, if we record a signal and then filter it, the "future" values of the signal are available. Thus, we can design better filters if the filters are to operate only on recorded signals; of course, the filtering is not performed in real time.

A system described by

$$y(t) = x(t - 2), \tag{2.67}$$

with t in seconds, is causal, since the present output is equal to the input of 2 s ago. For example, we can realize this system by recording the signal $x(t)$ on magnetic tape. The playback head is then placed 2 s downstream on the tape from the recording head. A system described by (2.67) is called an *ideal time delay*. The form of the signal is not altered; the signal is simply delayed.

A system described by

$$y(t) = x(t + 2) \tag{2.68}$$

is not causal, since, for example, the output at $t = 0$ is equal to the input at $t = 2$ s. This system is an *ideal time advance,* which is not physically realizable.

Another example of a causal system is illustrated in Figure 2.41. In this system, a time delay of 30 s is followed by a time advance of 25 s. Hence, the total system is causal and can be realized physically. However, the time-advance part of the system is not causal, but it can be realized if preceded by a time delay of at least 25 s. An example of this type of system is the non–real-time filtering described earlier.

Stability

We now define *stability*. Many definitions exist for the stability of a system; we give the *bounded-input–bounded-output* (BIBO) definition.

> **BIBO Stability**
> A system is stable if the output remains bounded for any bounded input.

By definition, a signal $x(t)$ is bounded if there exists a *number M* such that

$$|x(t)| \leqq M \text{ for all } t. \tag{2.69}$$

Hence, a system is bounded-input bounded-output stable if, for a *number R*,

$$|y(t)| \leqq R \text{ for all } t \tag{2.70}$$

for *all* $x(t)$ such that (2.69) is satisfied. Bounded $x(t)$ and $y(t)$ are illustrated in Figure 2.42. To determine BIBO stability for a given system, given any value M in (2.69), a value R (in general, a function of M) must be found such that (2.70) is satisfied.

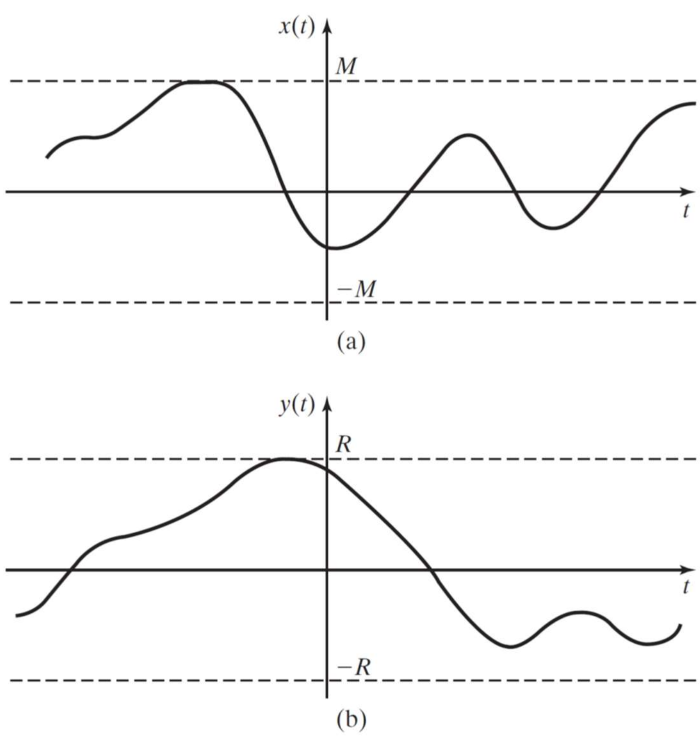


Figure 2.42 Bounded functions.

A squaring circuit

$$y(t) = x^2(t) \tag{2.71}$$

is stable because $y(t)$ is bounded for any bounded input. In (2.69) and (2.70),

$$|y(t)| \leq R = M^2.$$

Stability is a basic property required of almost all physical systems. Generally, a system that is not stable cannot be controlled and is of no value. An example of an unstable system is a public address system that has broken into oscillation; the output of this system is unrelated to its input. A second example of an unstable system has been seen several times in television news segments: the first stage of a space booster or a missile that went out of control (unstable) and had to be destroyed.

Time Invariance

A system is said to be time invariant if a time shift in the input signal results *only* in the same time shift in the output signal.

For a time-invariant system for which the input $x(t)$ produces the output $y(t)$, $y(t) = T[x(t)]$, $x(t - t_0)$ produces $y(t - t_0)$. That is,

$$y(t - t_0) = T[x(t - t_0)] \tag{2.73}$$

for all t_0, where $T[x(t - t_0)]$ indicates the transformation that describes the system's input–output relationship. In other words, a time-invariant system does not change with time; if it is used today, it will behave the same way as it would next week or next year. A time-invariant system is also called a *fixed system*.

A test for time invariance is illustrated in Figure 2.44. The signal $y(t - t_0)$ is obtained by delaying $y(t)$ by t_0 seconds. We define $y_d(t)$ as the system output for the delayed input $x(t - t_0)$, such that

$$y_d(t) = T[x(t - t_0)]$$

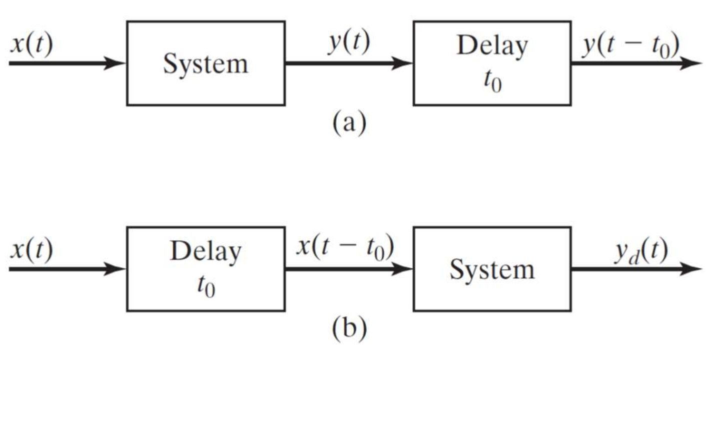


Figure 2.44 Test for time invariance.

The system in Figure 2.44 is time invariant, provided that

$$y(t - t_0) = y_d(t). \tag{2.74}$$

A system that is not time invariant is *time varying*.

As an example of time invariance, consider the system

$$y(t) = e^{x(t)}.$$

From (2.73) and (2.74),

$$y_d(t) = y(t)\Big|_{x(t-t_0),} = e^{x(t-t_0)} = y(t)\Big|_{t-t_0,}$$

and the system is time invariant.

Consider next the system

$$y(t) = e^{-t}x(t).$$

In (2.73) and (2.74),

$$y_d(t) = y(t)\Big|_{x(t-t_0)} = e^{-t}x(t - t_0)$$

and

$$y(t)\Big|_{t-t_0} = e^{-(t-t_0)}x(t - t_0).$$

The last two expressions are not equal; therefore, (2.74) is not satisfied, and the system is time varying.

EXAMPLE 2.18 **Test for time invariance**

Figure 2.45(a) illustrates the test for time invariance (2.73) for a system that performs a time reversal on the input signal. The input signal chosen for the test is a unit step function, $x(t) = u(t)$. In the top branch of Figure 2.45(a), we first reverse $u(t)$ to obtain $y(t) = u(-t)$ and then delay it by 1 s to form $y(t - 1) = u(-(t - 1)) = u(1 - t)$. In the bottom branch of the diagram, we first delay the input by 1 s to form $u(t - 1)$ and then reverse in time to form $y_d(t) = u(-t - 1)$. The signals for this system are shown in Figure 2.45(b). Because $y_d(t) \neq y(t - 1)$, the time-reversal operation is not time invariant. Intuitively, this makes sense because a time shift to the right before a time reversal will result in a time shift to the left after the time reversal. ■

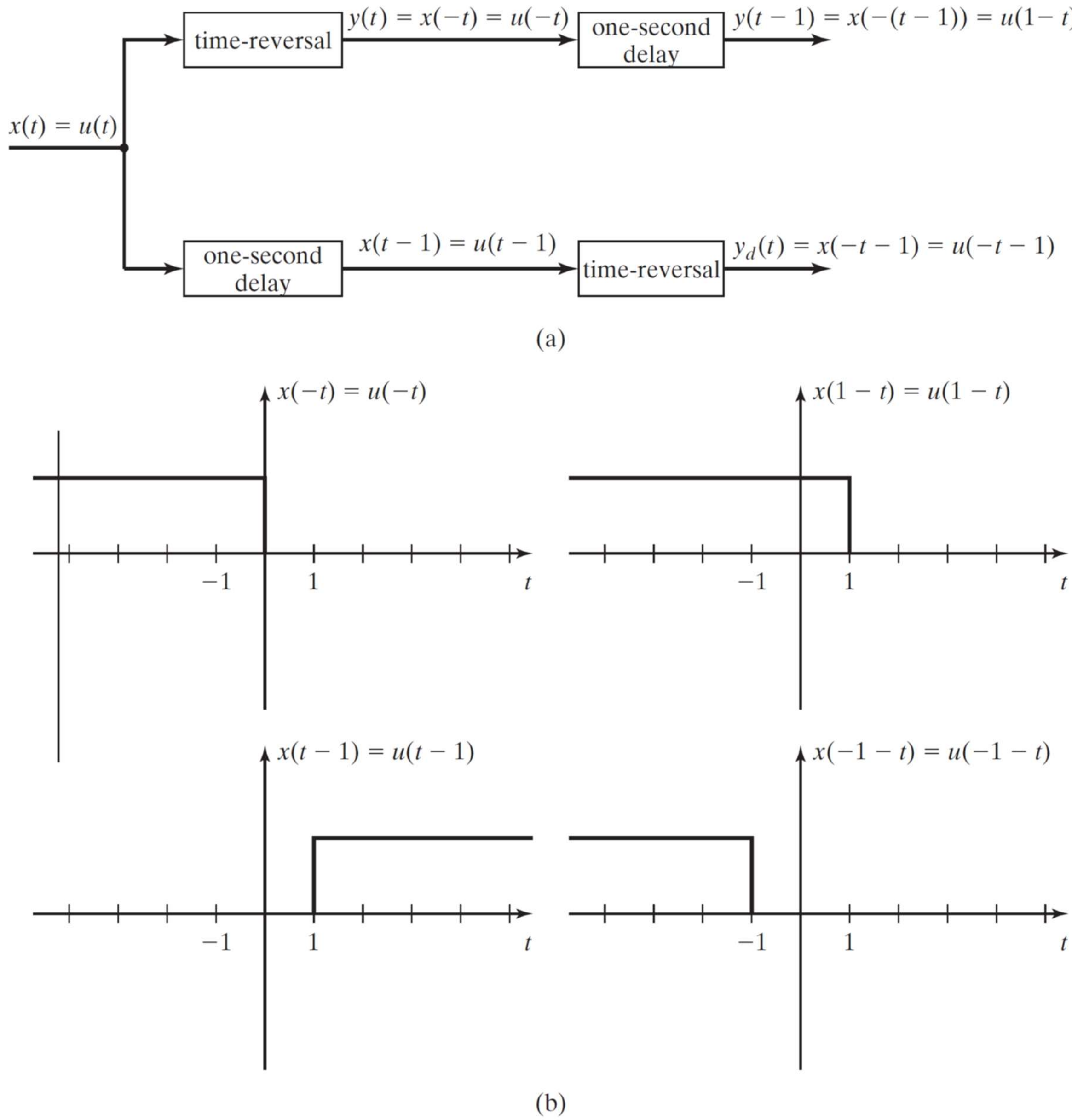


Figure 2.45 Time invariance test for Example 2.18.

Linearity

The property of *linearity* is one of the most important properties that we consider. Once again, we define the system input signal to be $x(t)$ and the output signal to be $y(t)$.

Linear System

A system is linear if it meets the following two criteria:

1. Additivity: If $x_1(t) \to y_1(t)$ and $x_2(t) \to y_2(t)$, then

$$x_1(t) + x_2(t) \to y_1(t) + y_2(t). \tag{2.75}$$

2. Homogeneity: If $x_1(t) \to y_1(t)$, then

$$ax_1(t) \to ay_1(t), \tag{2.76}$$

where a is a constant. The criteria must apply for all $x_1(t)$ and $x_2(t)$ and for all a.

These two criteria can be combined to yield the *principle of superposition*. A system satisfies the principle of superposition if, with the inputs and outputs as just defined,

$$a_1x_1(t) + a_2x_2(t) \to a_1y_1(t) + a_2y_2(t), \tag{2.77}$$

where a_1 and a_2 are constants. A system is linear if and only if it satisfies the principle of superposition.

No physical system is linear under all operating conditions. However, a physical system can be tested by using (2.77) to determine ranges of operation for which the system is approximately linear.

An example of a linear system is an ideal amplifier, described by $y(t) = Kx(t)$. An example of a nonlinear system is the squaring circuit mentioned earlier:

$$y(t) = x^2(t).$$

For inputs of $x_1(t)$ and $x_2(t)$, the outputs are

$$x_1(t) \to x_1^2(t) = y_1(t)$$

and

$$x_2(t) \to x_2^2(t) = y_2(t). \tag{2.78}$$

However, the input $[x_1(t) + x_2(t)]$ produces the output

$$x_1(t) + x_2(t) \rightarrow [x_1(t) + x_2(t)]^2 = x_1^2(t) + 2x_1(t)x_2(t) + x_2^2(t) = y_1(t) + y_2(t) + 2x_1(t)x_2(t), \quad (2.79)$$

and $[x_1(t) + x_2(t)]$ does not produce $[y_1(t) + y_2(t)]$. Hence, the squaring circuit is nonlinear.

A linear time-invariant (LTI) system is a linear system that is also time invariant. The LTI system, for both continuous-time and discrete-time systems, is the type that is emphasized in this book.

An important class of continuous-time LTI systems consists of those that can be modeled by linear differential equations with constant coefficients. An example of this type of system is the *RL* circuit of Figure 2.33, modeled by

[eq(2.53)]
$$L\frac{di(t)}{dt} + Ri(t) = v(t).$$

EXAMPLE 2.19 **Determining the properties of a particular system**

The characteristics for the system

$$y(t) = \sin(2t)\, x(t)$$

are now investigated. Note that this system can be considered to be an amplifier with a time-varying gain that varies between −1 and 1—that is, with the gain $K(t) = \sin 2t$ and $y(t) = K(t)x(t)$. The characteristics are as follows:

1. This system is *memoryless,* because the output is a function of the input at only the present time.
2. The system is *not invertible,* because, for example, $y(\pi) = 0$, regardless of the value of the input. Hence, the system has no inverse.

3. The system is *causal,* because the output does not depend on the input at a future time.
4. The system is *stable,* the output is bounded for all bounded inputs, because the multiplier sin (2*t*) has a maximum value of 1. If $|x(t)| \leqq M, |y(t)| \leqq M$, also.
5. The system is *time varying*. From (2.73) and (2.74),

$$y_d(t) = y(t)\Big|_{x(t-t_0)} = \sin 2t x(t - t_0)$$

and

$$y(t)\Big|_{t-t_0} = \sin 2(t - t_0)x(t - t_0).$$

6. The system is *linear,* since

$$a_1x_1(t) + a_2x_2(t) \rightarrow \sin 2t[a_1x_1(t) + a_2x_2(t)] = a_1 \sin 2tx_1(t) + a_2 \sin 2tx_2(t) = a_1y_1(t) + a_2y_2(t).$$

■

EXAMPLE 2.20 **Testing for linearity by using superposition**

As a final example, consider the system described by the equation $y(t) = 3x(t)$, a linear amplifier. This system is easily shown to be linear by the use of superposition. However, the system $y(t) = [3x(t) + 1.5]$, an amplifier that adds a dc component, is nonlinear. By superposition,

$$y(t) = 3[a_1x_1(t) + a_2x_2(t)] + 1.5 \neq a_1y_1(t) + a_2y_2(t).$$

This system is not linear, because a part of the output signal is independent of the input signal. ■

II.1. Summary

Equation Title	Equation Number	Equation
Independent-variable transformation	(2.6)	$y(t) = x(at + b)$
Signal-amplitude transformation	(2.8)	$y(t) = Ax(t) + B$
Even part of a signal	(2.13)	$x_e(t) = \frac{1}{2}[x(t) + x(-t)]$
Odd part of a signal	(2.14)	$x_o(t) = \frac{1}{2}[x(t) - x(-t)]$
Definition of periodicity	(2.15)	$x(t) = x(t + T), \quad T > 0$
Fundamental frequency in hertz and radians/second	(2.16)	$f_0 = \frac{1}{T_0}$ Hz, $\omega_0 = 2\pi f_0 = \frac{2\pi}{T_0}$ rad/s
Exponential function	(2.18)	$x(t) = Ce^{at}$
Euler's relation	(2.19)	$e^{j\theta} = \cos\theta + j\sin\theta$
Cosine equation	(2.21)	$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$
Sine equation	(2.22)	$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$

Complex exponential in polar form	(2.23) and (2.24)	$e^{j\theta} = 1\angle\theta$ and $\arg e^{j\theta} = \tan^{-1}\left[\frac{\sin\theta}{\cos\theta}\right] = \theta$
Unit step function	(2.32)	$u(\tau) = \begin{cases} 1, & \tau > 0 \\ 0, & \tau < 0 \end{cases}$
Unit impulse function	(2.40)	$\delta(t - t_0) = 0, \quad t \neq t_0;$ $\int_{-\infty}^{\infty} \delta(t - t_0)dt = 1$
Sifting property of unit impulse function	(2.41)	$\int_{-\infty}^{\infty} f(t)\delta(t - t_0)dt = f(t_0)$
Multiplication property of unit impulse function	(2.42)	$f(t)\delta(t - t_0) = f(t_0)\delta(t - t_0)$
Test for time invariance	(2.73)	$y(t)\Big\vert_{t-t_0} = y(t)\Big\vert_{x(t-t_0)}$
Test for linearity	(2.77)	$a_1x_1(t) + a_2x_2(t) \rightarrow a_1y_1(t) + a_2y_2(t)$

II.2. Continuous Linear Time Invariant Systems

Consider a system described by

$$x(t) \rightarrow y(t). \tag{3.1}$$

This system is *time invariant* if a time shift of the input results in the same time shift of the output—that is, if

$$x(t - t_0) \rightarrow y(t - t_0), \tag{3.2}$$

where t_0 is an arbitrary constant.

For the system of (3.1), let

$$x_1(t) \rightarrow y_1(t), \quad x_2(t) \rightarrow y_2(t). \tag{3.3}$$

This system is *linear,* provided that the principle of superposition applies:

$$a_1x_1(t) + a_2x_2(t) \rightarrow a_1y_1(t) + a_2y_2(t). \tag{3.4}$$

This property applies for all constants a_1 and a_2 and for all signals $x_1(t)$ and $x_2(t)$.

IMPULSE REPRESENTATION OF CONTINUOUS-TIME SIGNALS

In this section, a relationship is developed that expresses a general signal $x(t)$ as a function of an impulse function. This relationship is useful in deriving general properties of continuous-time linear time-invariant (LTI) systems.

Recall that two definitions of the impulse function are given in Section 2.4. The first definition is, from (2.40),

$$\int_{-\infty}^{\infty} \delta(t - t_0)\, dt = 1, \tag{3.5}$$

and the second one is, from (2.41),

$$\int_{-\infty}^{\infty} x(t)\delta(t - t_0)\, dt = x(t_0). \tag{3.6}$$

The second definition requires that $x(t)$ be continuous at $t = t_0$. According to (3.6), if $x(t)$ is continuous at $t = t_0$, the sifting property of impulse functions can be stated as from (2.42),

$$x(t)\delta(t - t_0) = x(t_0)\delta(t - t_0). \tag{3.7}$$

We now derive the desired relationship. From (3.7), with $t_0 = \tau$,

$$x(t)\delta(t - \tau) = x(\tau)\delta(t - \tau).$$

From (3.5), we use the preceding result to express $x(t)$ as an integral involving an impulse function:

$$\begin{aligned}\int_{-\infty}^{\infty} x(\tau)\delta(t - \tau)\, d\tau &= \int_{-\infty}^{\infty} x(t)\delta(t - \tau)\, d\tau \\ &= x(t)\int_{-\infty}^{\infty} \delta(t - \tau)\, d\tau = x(t).\end{aligned}$$

We rewrite this equation as

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t - \tau)\, d\tau. \tag{3.8}$$

This equation is the desired result, in which a general signal $x(t)$ is expressed as a function of an impulse function. We use this expression for $x(t)$ in the next section.

CONVOLUTION FOR CONTINUOUS-TIME LTI SYSTEMS

An equation relating the output of a continuous-time LTI system to its input is developed in this section. We begin the development by considering the system shown in Figure 3.1, for which

$$x(t) \rightarrow y(t).$$

A unit impulse function $\delta(t)$ is applied to the system input. Recall the description (3.5) of this input signal; the input signal is zero at all values of time other than $t = 0$, at which time the signal is unbounded.

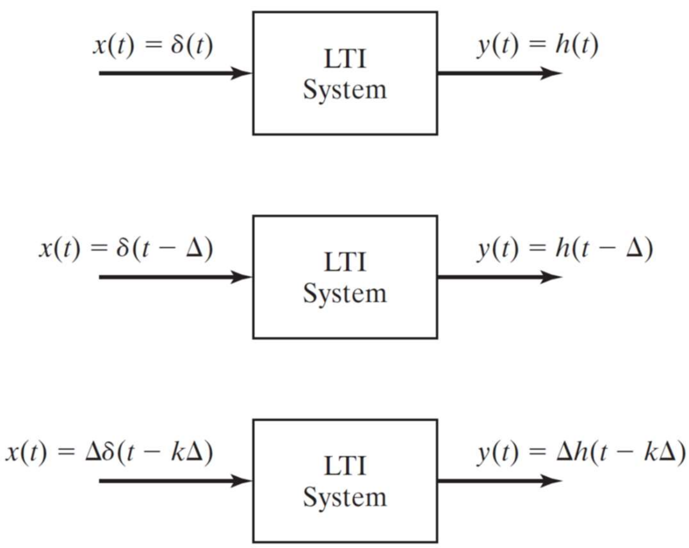


Figure 3.1 Impulse response of an LTI system.

With the input an impulse function, we denote the LTI system response in Figure 3.1 as $h(t)$; that is,

$$\delta(t) \rightarrow h(t). \tag{3.9}$$

Because the system is time invariant, the response to a time-shifted impulse function, $\delta(t - t_0)$, is given by

$$\delta(t - t_0) \rightarrow h(t - t_0).$$

The notation $h(\cdot)$ will *always* denote the *unit impulse response.*

According to the principle of superposition, an LTI system's total response to a sum of inputs is the sum of the responses to each individual input. It follows that if the input is a sum of weighted, time-shifted impulses

$$x(t) = \sum_{k=0}^{\infty} \Delta\delta(t - k\Delta), \tag{3.10}$$

then the output signal is a sum of weighted, time-shifted impulse responses

$$y(t) = \sum_{k=0}^{\infty} \Delta h(t - k\Delta). \tag{3.11}$$

EXAMPLE 3.1 **Sum of impulse responses**

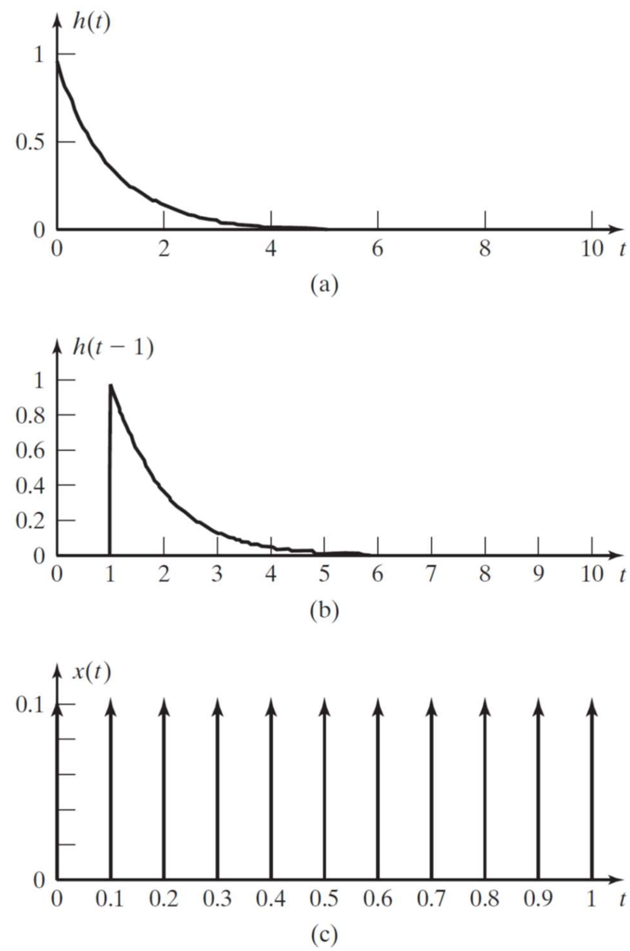
Consider the system with the impulse response $h(t) = e^{-t}u(t)$, as shown in Figure 3.2(a). This system's response to an input of $x(t) = \delta(t - 1)$ would be $y(t) = h(t - 1) = e^{-(t-1)}u(t - 1)$, as shown in Figure 3.2(b). If the input signal is a sum of weighted, time-shifted impulses as described by (3.10), separated in time by $\Delta = 0.1$ (s) so that

$$x(t) = \sum_{k=0}^{\infty} 0.1\delta(t - 0.1k),$$

as shown in Figure 3.2(c), then, according to (3.11), the output is

$$y(t) = \sum_{k=0}^{\infty} 0.1h(t - 0.1k) = 0.1\sum_{k=0}^{\infty} e^{-(t-0.1k)}u(t - 0.1k).$$

This output signal is plotted in Figure 3.2(d).



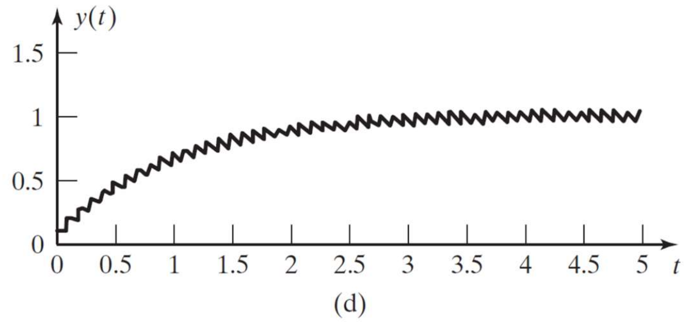


Figure 3.2 Impulse responses for the system of Example 3.1.

So we can now write

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau, \tag{3.13}$$

The result in (3.13) is called the *convolution integral*. We denote this integral with an asterisk, as in the following notation:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau)\, d\tau = x(t)*h(t). \tag{3.14}$$

Next we derive an important property of the convolution integral by making a change of variables in (3.13); let $s = (t - \tau)$. Then $\tau = (t - s)$ and $d\tau = -ds$. Equation (3.13) becomes

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} x(\tau)h(t - \tau)\, d\tau = \int_{\infty}^{-\infty} x(t - s)h(s)[-ds] \\ &= \int_{-\infty}^{\infty} x(t - s)h(s)\, ds. \end{aligned}$$

Next we replace s with τ in the last integral, and thus the convolution can also be expressed as

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau)\, d\tau = \int_{-\infty}^{\infty} x(t - \tau)h(\tau)\, d\tau. \tag{3.15}$$

The convolution integral is symmetrical with respect to the input signal $x(t)$ and the impulse response $h(t)$, and we have the property

$$y(t) = x(t)*h(t) = h(t)*x(t). \tag{3.16}$$

We derive an additional property of the convolution integral by considering the convolution integral for a unit impulse input; that is, for $x(t) = \delta(t)$,

$$y(t) = \delta(t)*h(t).$$

This property is independent of the functional form of $h(t)$. Hence, the convolution of any function $g(t)$ with the unit impulse function yields that function $g(t)$. Because of the time-invariance property, the general form of (3.17) is given by

$$y(t - t_0) = \delta(t - t_0)*h(t) = h(t - t_0).$$

This general property may be stated in terms of a function $g(t)$ as

$$\delta(t)*g(t) = g(t)$$

and (3.18)

$$\delta(t - t_0)*g(t) = g(t - t_0)*\delta(t) = g(t - t_0).$$

The second relationship is based on (3.16).

EXAMPLE 3.2 **Impulse response of an integrator**

Consider the system of Figure 3.3. The system is an integrator, in which the output is the integral of the input:

$$y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau. \tag{3.20}$$

This equation is the mathematical model of the system. We use the integral symbol in a block to denote the integrator. The system is practical and can be realized as an electronic circuit with an operational amplifier, a resistor, and a capacitor, as described in Section 1.2. Integrating amplifiers of this type are used extensively in analog signal processing and in closed-loop control systems.

We see that the impulse response of this system is the integral of the unit impulse function, which is the unit step function:

$$h(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau = u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}.$$

$$y(t)=\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$
$$x(t)=tu(t)$$
$$x(\tau)=\tau u(\tau)$$
$$h(t)=u(t)$$
$$h(t-\tau)=u(t-\tau)$$

We will now use the convolution integral to find the system response for the unit ramp input, $x(t) = tu(t)$. From (3.14),

$$y(t) = x(t)*h(t) = tu(t)*u(t) = \int_{-\infty}^{\infty} \tau u(\tau)u(t - \tau)\, d\tau.$$

In this integral, t is considered to be constant. The unit step $u(\tau)$ is zero for $\tau < 0$; hence, the lower limit on the integral can be increased to zero with $u(\tau)$ removed from the integrand:

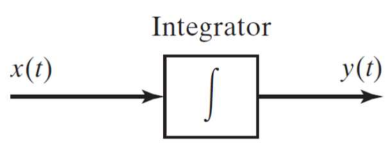
$$y(t) = \int_{0}^{\infty} \tau u(t - \tau)\, d\tau.$$

In addition, the unit step $u(t - \tau)$ is defined as

$$u(t - \tau) = \begin{cases} 0, & \tau > t \\ 1, & \tau < t \end{cases}.$$

The upper limit on the integral can then be reduced to t, and thus,

$$y(t) = \int_{0}^{t} \tau\, d\tau = \frac{\tau^2}{2}\bigg|_0^t = \frac{t^2}{2}u(t).$$



$$y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$$

Figure 3.3 System for Example 3.2.

This result is easily verified from the system equation, (3.20):

$$y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau = \int_{-\infty}^{t} \tau u(\tau)\, d\tau = \int_{0}^{t} \tau\, d\tau = \frac{t^2}{2}u(t). \quad ■$$

EXAMPLE 3.3 **Convolution for the system of Example 3.1**

Consider the system of Example 3.1 for the case that the time increment between impulses approaches zero. If we let $\Delta = 0.1$, the input signal in Example 3.1 can be written as

$$x(t) = \sum_{k=0}^{\infty} 0.1\delta(t - 0.1k) = \sum_{k=-\infty}^{\infty} u(k\Delta)\delta(t - k\Delta)\,(\Delta),$$

because $u(k\Delta) = 0$ for $k < 0$ and $u(k\Delta) = 1$ for $k \geq 1$. From (3.12), as $\Delta \to 0$, the input signal becomes

$$x(t) = \int_{-\infty}^{\infty} u(\tau)\delta(t - \tau)d\tau = u(t).$$

From (3.18), the output signal is calculated from

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau = \int_{-\infty}^{\infty} u(\tau)e^{-(t-\tau)}u(t - \tau)d\tau.$$

$$x(t) = u(t)$$

$$h(t) = e^{-t}u(t)$$

Because $u(\tau) = 0$ for $\tau < 0$, and $u(t - \tau) = 0$ for $\tau > t$, this convolution integral can be rewritten as

$$y(t) = \int_{0}^{t} e^{-(t-\tau)}d\tau = e^{-t}\int_{0}^{t} e^{\tau}d\tau = (1 - e^{-t})u(t).$$

The output signal is plotted in Figure 3.4. Compare this result with the summation result shown in Figure 3.2(d).

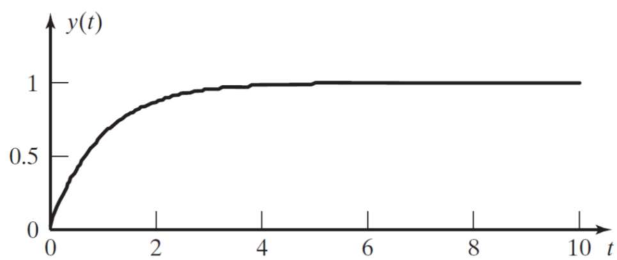


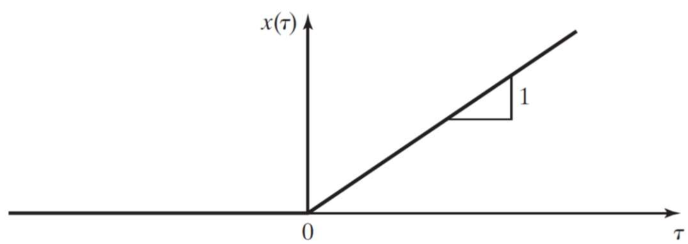
Figure 3.4 System output signal for Example 3.3. ■

EXAMPLE 3.4 **Graphical evaluation for the response of an integrator**

This example is a continuation of Example 3.2. The output for that system (an integrator) will be found graphically. The graphical solution will indicate some of the important properties of convolution. Recall from Example 3.2 that the impulse response of the system is the unit step function; that is, $h(t) = u(t)$. To find the system output, we evaluate the convolution integral

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau)\, d\tau.$$

Note that the integration is with respect to τ; hence, t is considered to be constant. Note also that the impulse response is time reversed to yield $h(-\tau)$, and then time shifted to yield $h(t - \tau)$. These signal manipulations are illustrated in Figure 3.5.



(a)

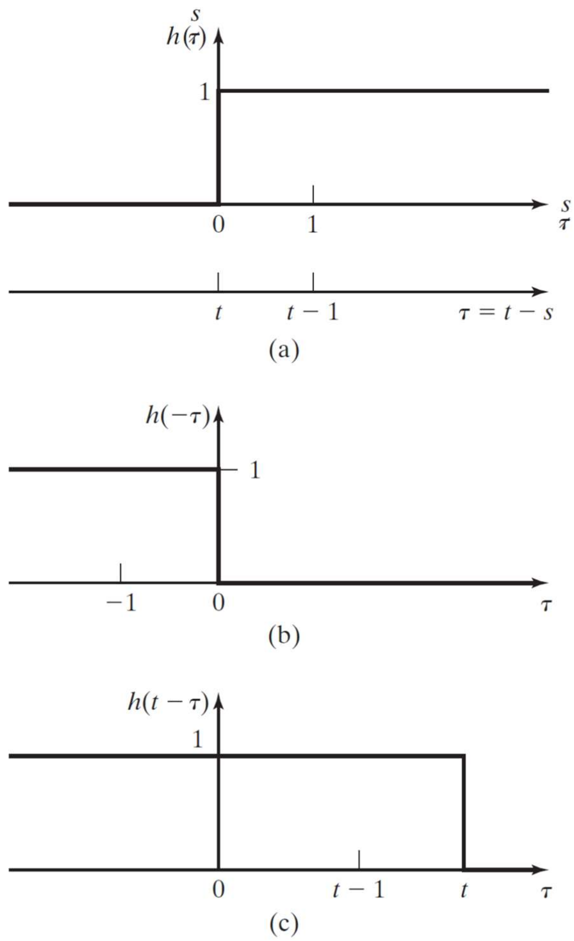


Figure 3.5 Impulse response factor for convolution.

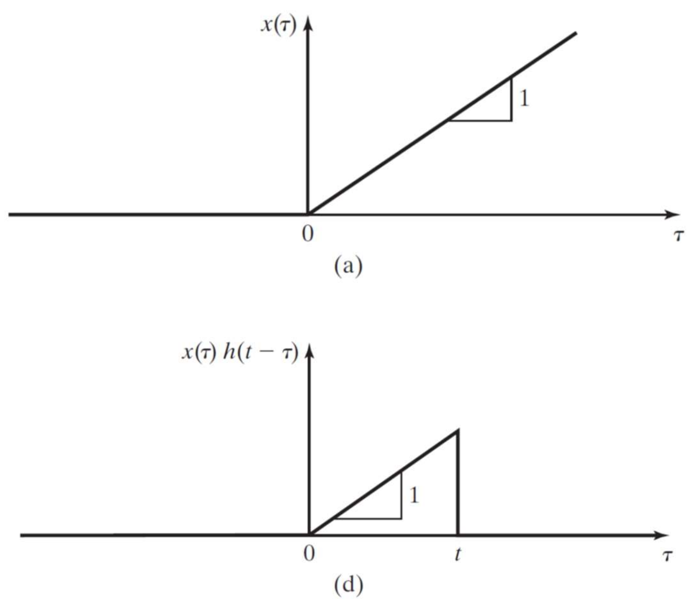


Figure 3.6 Convolution for Example 3.4.

The τ-axis is plotted in Figure 3.5(a) and is the same as the axis of $h(t-\tau)$ in Figure 3.5(c).

Shown in Figure 3.6(a) is the first term of the convolution integral, $x(\tau) = \tau u(\tau)$. Figure 3.6(b) shows the second term of the integral, $h(t-\tau)$, for $t < 0$. The product of these two functions is zero; hence, the value of the integral [and of $y(t)$] is zero for $t < 0$. Figure 3.6(c) shows the second term of the convolution integral for $t > 0$, and Figure 3.6(d) shows the product of the functions, $x(\tau)h(t-\tau)$, of Figure 3.6(a) and (c). Therefore, from the convolution integral, $y(t)$ is the area under the function in Figure 3.6(d). Because the product function is triangular, the area is equal to one-half the base times the height:

$$y(t) = \frac{1}{2}(t)(t) = \frac{t^2}{2}, \quad t > 0.$$

Example 1

$$x(t) = \begin{cases} 1, & 0 \le t \le T_1 \\ 0, & \text{otherwise} \end{cases} \Rightarrow x(\tau) = \begin{cases} 1, & 0 \le \tau \le T_1 \\ 0, & \text{otherwise} \end{cases}$$

$$h(t) = \begin{cases} 1, & 0 \le t \le T_2 \\ 0, & \text{otherwise} \end{cases} \Rightarrow h(t-\tau) = \begin{cases} 1, & 0 \le t-\tau \le T \Rightarrow -t \le -\tau \le -t+T \Rightarrow t-T \le \tau \le t \\ 0, & \text{otherwise} \end{cases}$$

$$T_1 = T_2 = T$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

$$= \int_0^T h(t-\tau)d\tau$$

For $t < 0$, $y(t) = 0$

For $t > 2T$, $y(t) = 0$

For $0 \le t \le T$, integrate from 0 to t to get

$$y(t) = \int_0^t d\tau = t$$

For $T \le t \le 2T$, integrate from t to T to get

$$y(t) = \int_{t-T}^{T} d\tau = 2T - t$$

Sketch $y(t)$.

EXAMPLE 3.5 **A system with a rectangular impulse response**

As a third example, let the impulse response for an LTI system be rectangular, as shown in Figure 3.7. We will later use this system in the study of certain types of sampling of continuous-time signals. Furthermore, this system is used in the modeling of digital-to-analog converters. Note that one realization of this system consists of an integrator, an ideal time delay, and a summing junction, as shown in Figure 3.8. The reader can show that the impulse response of the system is the rectangular pulse of Figure 3.7.

The input to this system is specified as

$$\begin{aligned} x(t) &= \delta(t+3) + 3e^{-0.5t}u(t) \\ &= x_1(t) + x_2(t) \end{aligned}$$

and is also plotted in Figure 3.7. We have expressed the input as the sum of two functions; by the linearity property, the response is the sum of the responses to each input function. Hence,

$$\left.\begin{aligned} x_1(t) &\to y_1(t) \\ x_2(t) &\to y_2(t) \end{aligned}\right\}, \quad x(t) \to y_1(t) + y_2(t).$$

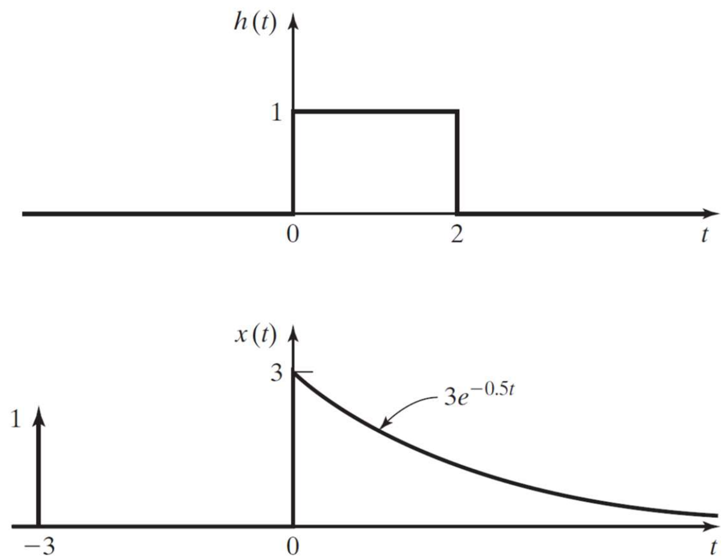


Figure 3.7 Input signal and impulse response for Example 3.5.

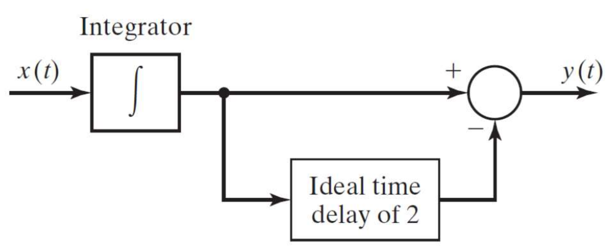


Figure 3.8 System for Example 3.5.

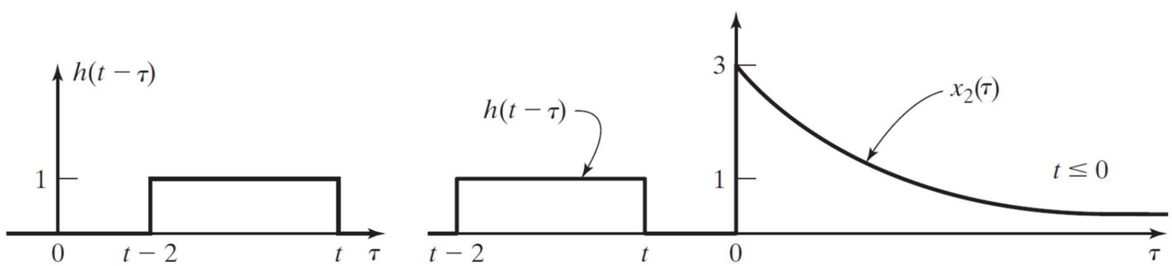
The system response to the impulse function is obtained from (3.18):

$$y_1(t) = h(t)*\delta(t+3) = h(t+3) = u(t+3) - u(t+1).$$

To determine the response to $x_2(t)$, we plot $h(t-\tau)$, as shown in Figure 3.9(a). This plot is obtained by time reversal and time shifting. Three different integrations must be performed to evaluate the convolution integral:

1. The first integration applies for $t \leqq 0$, as shown in Figure 3.9(a), and is given by

$$y_2(t) = \int_{-\infty}^{\infty} x_2(\tau)h(t-\tau)\,d\tau = \int_{-\infty}^{0} (0)h(t-\tau)\,d\tau + \int_{0}^{\infty} x_2(\tau)\,(0)\,d\tau = 0, \quad t \leqq 0.$$

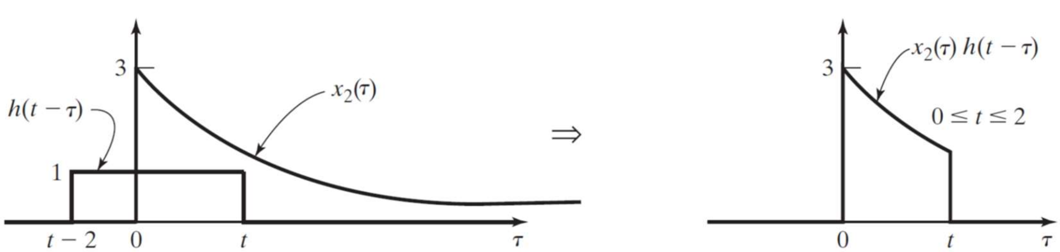


(a)

2. The second integration applies for $0 \leqq t \leqq 2$ and is illustrated in Figure 3.9(b):

$$y_2(t) = \int_{-\infty}^{\infty} x_2(\tau)h(t-\tau)\,d\tau = \int_{-\infty}^{0} (0)h(t-\tau)\,d\tau + \int_{0}^{t} 3e^{-0.5\tau}d\tau + \int_{t}^{\infty} x_2(\tau)(0)\,d\tau$$

$$= \frac{3e^{-0.5\tau}}{-0.5}\bigg|_0^t = 6(1 - e^{-0.5t}), \quad 0 \leqq t \leqq 2.$$



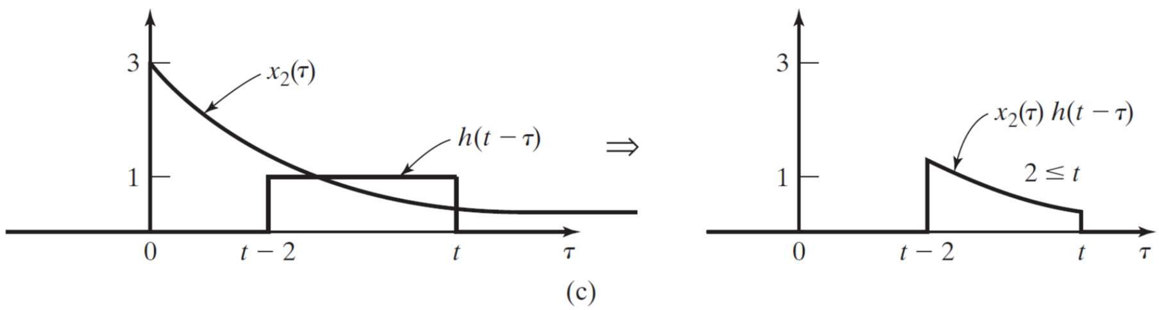
(b)

3. Figure 3.9(c) applies for $2 \leqq t \leqq \infty$:

$$y_2(t) = \int_{t-2}^{t} 3e^{-0.5\tau} d\tau = \frac{3e^{-0.5\tau}}{-0.5}\Bigg|_{t-2}^{t} = 6(e^{-0.5(t-2)} - e^{-0.5t})$$

$$= 6e^{-0.5t}(e^1 - 1) = 10.31e^{-0.5t}, \quad 2 \leqq t < \infty.$$

The output $y(t)$ is plotted in Figure 3.10.



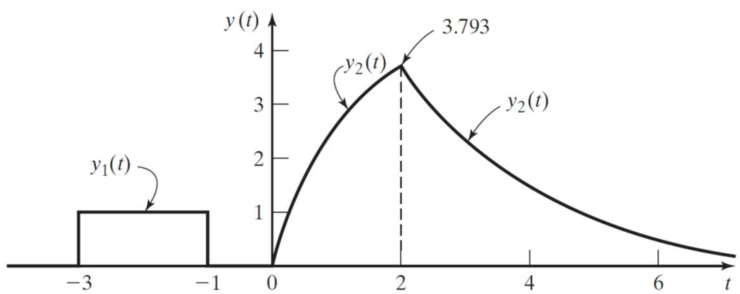
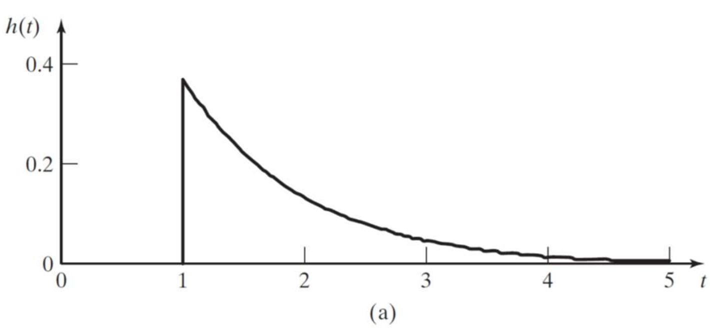
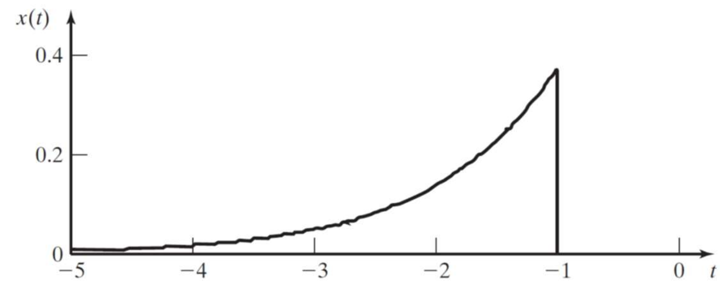


Figure 3.10 Output signal for Example 3.5. ■

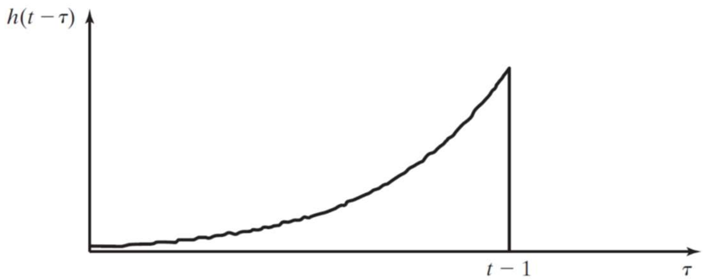
EXAMPLE 3.6 **A system with a time-delayed exponential impulse response**

Consider a system with an impulse response of $h(t) = e^{-t}u(t-1)$ and an input signal $x(t) = e^{t}u(-1-t)$. The system's impulse response and the input signal are shown in Figure 3.11(a) and (b), respectively. The system's output is $y(t) = x(t)^{*}h(t)$, from (3.16).

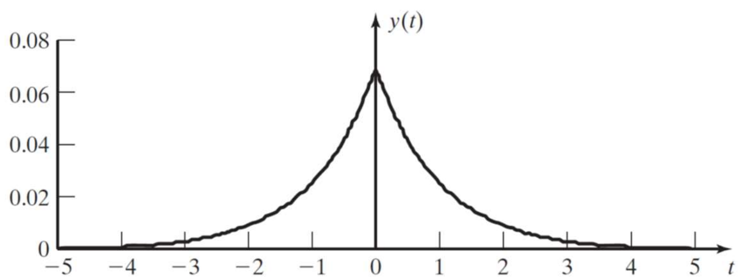




(b)



(c)



(d)

$$y(t) = \int_{-\infty}^{t-1} e^{\tau} e^{-(t-\tau)} d\tau = \int_{-\infty}^{t-1} e^{-t} e^{2\tau} d\tau = \frac{e^{-2} e^{t}}{2}, -\infty < t \leq 0.$$

Because $x(\tau)$ is zero for $\tau > -1$, for $t > 0$, the output signal is given by

$$y(t) = \int_{-\infty}^{-1} e^{\tau} e^{-(t-\tau)} d\tau = e^{-t} \int_{-\infty}^{-1} e^{2\tau} d\tau = \frac{e^{-2} e^{-t}}{2}, t > 0.$$

3.3 PROPERTIES OF CONVOLUTION

The convolution integral of (3.14) has three important properties:

1. *Commutative property*. As stated in (3.15), the convolution integral is symmetrical with respect to $x(t)$ and $h(t)$:

$$x(t)*h(t) = h(t)*x(t). \tag{3.22}$$

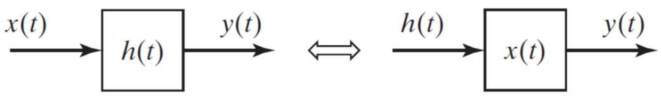


Figure 3.12 Commutative property.

2. *Associative property*. The result of the convolution of three or more functions is independent of the order in which the convolution is performed. For example,

$$[x(t)*h_1(t)]*h_2(t) = x(t)*[h_1(t)*h_2(t)] = x(t)*[h_2(t)*h_1(t)]. \tag{3.23}$$

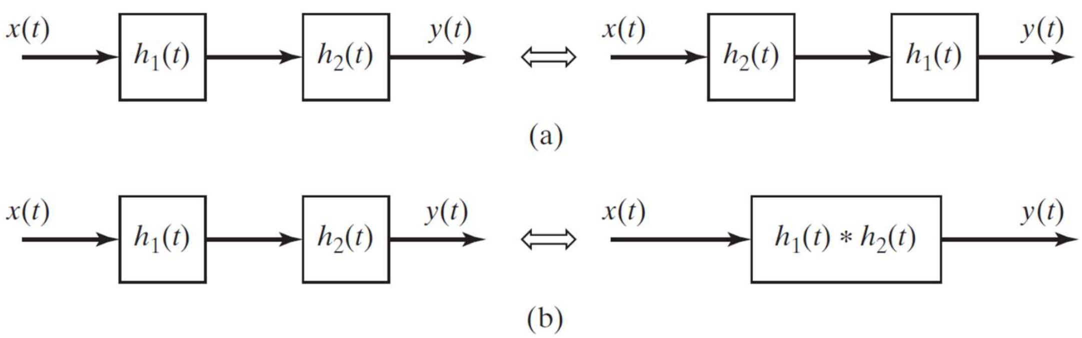


Figure 3.13 Associative property.

It follows that for m cascaded systems, the impulse response of the total system is given by

$$h(t) = h_1(t)*h_2(t)*\cdots*h_m(t). \tag{3.25}$$

3. *Distributive property*. The convolution integral satisfies the following relationship:

$$x(t)*h_1(t) + x(t)*h_2(t) = x(t)*[h_1(t) + h_2(t)]. \tag{3.26}$$

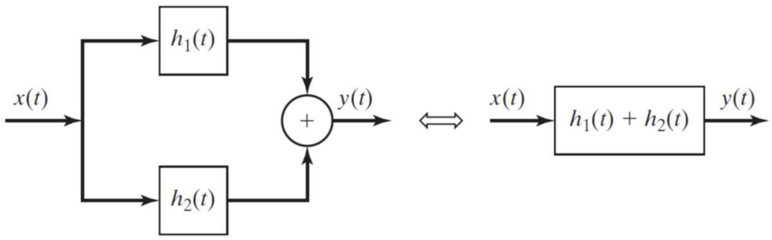


Figure 3.14 Distributive property.

EXAMPLE 3.7 **Impulse response for an interconnection of systems**

We wish to determine the impulse response of the system of Figure 3.15(a) in terms of the impulse responses of the four subsystems. First, from (3.29), the impulse response of the parallel systems 1 and 2 is given by

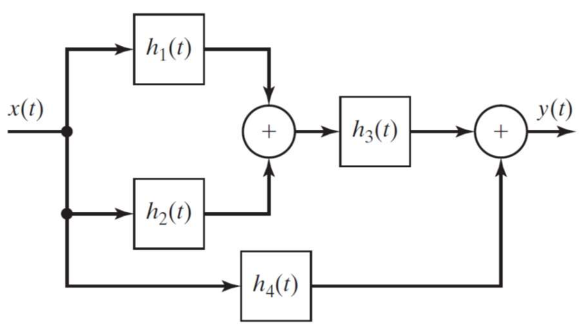
$$h_a(t) = h_1(t) + h_2(t),$$

as shown in Figure 3.15(b). From (3.24), the effect of the cascaded connection of system a and system 3 is given by

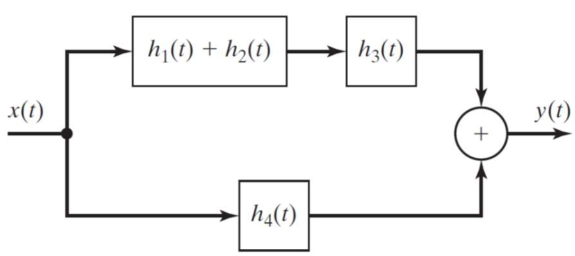
$$h_b(t) = h_a(t)*h_3(t) = [h_1(t) + h_2(t)]*h_3(t),$$

as shown in Figure 3.15(c). We add the effect of the parallel system 4 to give the total-system impulse response, as shown in Figure 3.15(d):

$$h(t) = h_b(t) + h_4(t) = [h_1(t) + h_2(t)]*h_3(t) + h_4(t).$$ ■



(a)



(b)

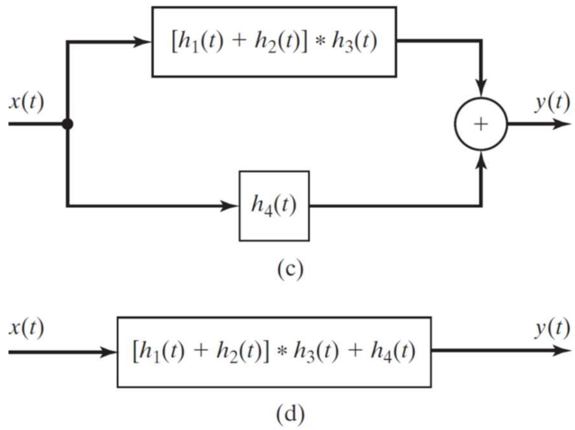


Figure 3.15 System for Example 3.7.

3.4 PROPERTIES OF CONTINUOUS-TIME LTI SYSTEMS

Memoryless Systems

Recall that a memoryless (static) system is one whose current value of output depends only on the current value of input; that is, the current value of the output does not depend on either past values or future values of the input. Let the present time be t_1. From (3.30),

$$y(t_1) = \int_{-\infty}^{\infty} x(\tau)h(t_1 - \tau)\, d\tau. \tag{3.31}$$

Hence, an LTI system is memoryless if and only if $h(t) = K\delta(t)$,—that is, if $y(t) = Kx(t)$. A memoryless LTI system can be considered to be an ideal amplifier, with $y(t) = Kx(t)$. If the gain is unity ($K = 1$), the identity system results.

Invertibility

A continuous-time LTI system with the impulse response $h(t)$ is invertible if its input can be determined from its output. An invertible LTI system is depicted in Figure 3.16. For this system,

$$x(t)*h(t)*h_i(t) = x(t), \tag{3.32}$$

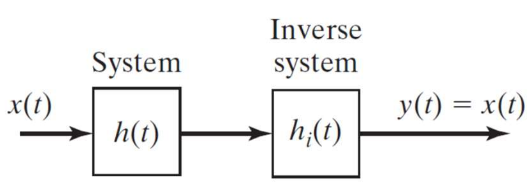


Figure 3.16 LTI invertible system.

$$x(t)*\delta(t) = x(t). \tag{3.33}$$

From (3.32) and (3.33),

$$h(t)*h_i(t) = \delta(t). \tag{3.34}$$

Thus, an LTI system with the impulse response $h(t)$ is invertible only if the function $h_i(t)$ can be found that satisfies (3.34). Then, the inverse system has the impulse response $h_i(t)$.

Causality

A continuous-time LTI system is causal if the current value of the output depends on only the current and past values of the input. Because the unit impulse function $\delta(t)$ occurs at $t = 0$, the impulse response $h(t)$ of a causal system *must be zero* for $t < 0$. In addition, a signal that is zero for $t < 0$ is called a *causal signal*. The convolution integral for a causal LTI system can then be expressed as

$$y(t) = \int_{-\infty}^{\infty} x(t - \tau)h(\tau)\,d\tau = \int_{0}^{\infty} x(t - \tau)h(\tau)\,d\tau. \tag{3.35}$$

If the impulse response is expressed as $h(t - \tau)$, this function is zero for $(t - \tau) < 0$, or for $\tau > t$. The second form of the convolution integral can then be expressed as

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau)\,d\tau = \int_{-\infty}^{t} x(\tau)h(t - \tau)\,d\tau. \tag{3.36}$$

Notice that (3.36) makes it clear that for a causal system, the output, $y(t)$, depends on values of the input only up to the present time, t, and not on future inputs.

In summary, for a causal continuous-time LTI system, the convolution integral can be expressed in the two forms

$$y(t) = \int_{0}^{\infty} x(t - \tau)h(\tau)\,d\tau = \int_{-\infty}^{t} x(\tau)h(t - \tau)\,d\tau. \tag{3.37}$$

Stability

Recall that a system is bounded-input–bounded-output (BIBO) stable if the output remains bounded for any bounded input. The boundedness of the input can be expressed as

$$|x(t)| < M \quad \text{for all } t,$$

where M is a real constant. Then, from (3.15), we can write

$$\begin{aligned}|y(t)| &= \left|\int_{-\infty}^{\infty} x(t-\tau)h(\tau)\,d\tau\right| \leqq \int_{-\infty}^{\infty} |x(t-\tau)h(\tau)|\,d\tau \\ &= \int_{-\infty}^{\infty} |x(t-\tau)||h(\tau)|\,d\tau \qquad (3.38)\\ &\leqq \int_{-\infty}^{\infty} M|h(\tau)|\,d\tau = M\int_{-\infty}^{\infty}|h(\tau)|\,d\tau,\end{aligned}$$

since

$$\left|\int_{-\infty}^{\infty} x_1(t)x_2(t)\,dt\right| \leqq \int_{-\infty}^{\infty}|x_1(t)x_2(t)|\,dt.$$

Thus, because M is finite, $y(t)$ is bounded if

$$\int_{-\infty}^{\infty}|h(t)|\,dt < \infty. \qquad (3.39)$$

EXAMPLE 3.8 **Stability for an LTI system derived**

We will determine the stability of the causal LTI system that has the impulse response given by

$$h(t) = e^{-3t}u(t).$$

In (3.40),

$$\int_{-\infty}^{\infty}|h(t)|\,dt = \int_{0}^{\infty} e^{-3t}\,dt = \left.\frac{e^{-3t}}{-3}\right|_0^{\infty} = \frac{1}{3} < \infty,$$

EXAMPLE 3.9 **Stability for an integrator examined**

As a second example, consider an LTI system such that $h(t) = u(t)$. From Example 3.2, this system is an integrator, with the output equal to the integral of the input:

$$y(t) = \int_{-\infty}^{t} x(\tau)\,d\tau.$$

We determine the stability of this system from (3.40), since the system is causal:

$$\int_{0}^{\infty}|h(t)|\,dt = \int_{0}^{\infty} dt = t\Big|_0^{\infty}.$$

This function is unbounded, and thus the system is not BIBO stable. ■

Unit Step Response

As has been stated several times, the impulse response of a system, $h(t)$, completely specifies the input–output characteristics of that system; the convolution integral,

[eq(3.30)] $$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau,$$

allows the calculation of the output signal $y(t)$ for any input signal $x(t)$.

Suppose that the system input is the unit step function, $u(t)$. From (3.30), with $s(t)$ denoting the unit step response, we obtain

$$s(t) = \int_{-\infty}^{\infty} u(\tau)h(t-\tau)\,d\tau = \int_{0}^{\infty} h(t-\tau)\,d\tau, \tag{3.41}$$

because $u(\tau)$ is zero for $\tau < 0$. If the system is causal, $h(t-\tau)$ is zero for $(t-\tau) < 0$, or for $\tau > t$, and

$$s(t) = \int_{0}^{t} h(\tau)\,d\tau. \tag{3.42}$$

We see, then, that the unit step response can be calculated directly from the unit impulse response, with the use of either (3.41) or (3.42).

If (3.41) or (3.42) is differentiated (see Leibnitz's rule, Appendix B), we obtain

$$h(t) = \frac{ds(t)}{dt}. \tag{3.43}$$

Thus, the unit impulse response can be calculated directly from the unit step response, and we see that the unit step response also completely describes the input–output characteristics of an LTI system.

EXAMPLE 3.10 **Step response from the impulse response**

Consider again the system of Example 3.8, which has the impulse response given by

$$h(t) = e^{-3t}u(t).$$

Note that this system is causal. From (3.42), the unit step response is then

$$s(t) = \int_0^t h(\tau)\,d\tau = \int_0^t e^{-3\tau}\,d\tau = \frac{e^{-3t}}{-3}\bigg|_0^t = \frac{1}{3}(1 - e^{-3t})u(t).$$

We can verify this result by differentiating $s(t)$ to obtain the impulse response. From (3.43), we get

$$\begin{aligned} h(t) &= \frac{ds(t)}{dt} = \frac{1}{3}(1 - e^{-3t})\delta(t) + \frac{1}{3}(-e^{-3t})(-3)u(t) \\ &= e^{-3t}u(t). \end{aligned}$$

Why does $(1 - e^{-3t})\delta(t) = 0$? [See (2.42).] ■

3.5 DIFFERENTIAL-EQUATION MODELS

Some properties of LTI continuous-time systems were developed in earlier sections of this chapter, with little reference to the actual equations that are used to model these systems. We now consider the most common model for LTI systems. Continuous-time LTI systems are usually modeled by *ordinary linear differential equations with constant coefficients*. We emphasize that we are considering the *models* of physical systems, not the physical systems themselves.

In Section 2.3, we considered the system model given by

$$\frac{dy(t)}{dt} = ay(t), \tag{3.44}$$

where a is constant. The system input $x(t)$ usually enters this model in the form

$$\frac{dy(t)}{dt} - ay(t) = bx(t), \tag{3.45}$$

where a and b are constants and $y(t)$ is the system output signal. The *order* of the system is the order of the differential equation that models the system. Hence, (3.45) is a *first-order system.*

Equation (3.45) is an ordinary linear differential equation with constant coefficients. The equation is ordinary, since no partial derivatives are involved. The equation is linear, since the equation contains the dependent variable and its derivative to the first degree only. One of the coefficients in the equation is equal to unity, one is $-a$, and one is b; hence, the equation has constant coefficients.

We now test the linearity of (3.45), using superposition. Let $y_i(t)$ denote the solution of (3.45) for the excitation $x_i(t)$, for $i = 1, 2$. By this, we mean that

$$\frac{dy_i(t)}{dt} - ay_i(t) = bx_i(t), \quad i = 1, 2. \tag{3.46}$$

We now show that the solution $[a_1y_1(t) + a_2y_2(t)]$ satisfies (3.45) for the excitation $[a_1x_1(t) + a_2x_2(t)]$, by direct substitution into (3.45):

$$\frac{d}{dt}[a_1y_1(t) + a_2y_2(t)] - a[a_1y_1(t) + a_2y_2(t)] = b[a_1x_1(t) + a_2x_2(t)].$$

This equation is rearranged to yield

$$a_1\left[\frac{dy_1(t)}{dt} - ay_1(t) - bx_1(t)\right] + a_2\left[\frac{dy_2(t)}{dt} - ay_2(t) - bx_2(t)\right] = 0. \tag{3.47}$$

Because, from (3.46), each term is equal to zero, the differential equation satisfies the principle of superposition and hence is linear.

Next, we test the model for time invariance. In (3.45), replacing t with $(t - t_0)$ results in the equation

$$\frac{dy(t - t_0)}{dt} - ay(t - t_0) = bx(t - t_0). \tag{3.48}$$

Delaying the input by t_0 delays the solution by the same amount; this system is then time invariant.

A simple example of an ordinary linear differential equation with constant coefficients is the first-order differential equation

$$\frac{dy(t)}{dt} + 2y(t) = x(t). \tag{3.49}$$

This equation could model the circuit of Figure 3.17, namely,

$$L\frac{di(t)}{dt} + Ri(t) = v(t),$$

with $L = 1\text{H}$, $R = 2\Omega$, $y(t) = i(t)$, and $x(t) = v(t)$.

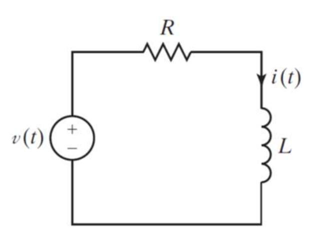


Figure 3.17 RL circuit.

The general form of an nth-order linear differential equation with constant coefficients is

$$a_n \frac{d^n y(t)}{dt^n} + a_{n-1} \frac{d^{n-1} y(t)}{dt^{n-1}} + \cdots + a_1 \frac{dy(t)}{dt} + a_0 y(t)$$
$$= b_m \frac{d^m x(t)}{dt^m} + b_{m-1} \frac{d^{m-1} x(t)}{dt^{m-1}} + \cdots + b_1 \frac{dx(t)}{dt} + b_0 x(t),$$

where $a_0, \ldots, a_n$ and $b_0, \ldots, b_m$ are constants and $a_n \neq 0$. We limit these constants to having real values. This equation can be expressed in the more compact form

$$\sum_{k=0}^{n} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{m} b_k \frac{d^k x(t)}{dt^k}. \tag{3.50}$$

Solution of Differential Equations

The method of solution of (3.50) presented here is called the *method of undetermined coefficients* [2] and requires that the general solution $y(t)$ be expressed as the sum of two functions:

$$y(t) = y_c(t) + y_p(t). \tag{3.51}$$

In this equation, $y_c(t)$ is called the *complementary function* and $y_p(t)$ is a *particular solution*. For the case that the differential equation models a system, the complementary function is usually called the *natural response*, and the particular solution, the *forced response*. We will use this notation. We only outline the method of solution; this method is presented in greater detail in Appendix E for readers requiring more review. The solution procedure is given as three steps:

EXAMPLE 3.11 **System response for a first-order LTI system**

As an example, we consider the differential equation given earlier in the section, but with $x(t)$ constant; that is,

$$\frac{dy(t)}{dt} + 2y(t) = 2$$

for $t \geqq 0$, with $y(0) = 4$. In Step 1, we assume the natural response $y_c(t) = Ce^{st}$. Then we substitute $y_c(t)$ into the homogeneous equation:

$$\frac{dy_c(t)}{dt} + 2y_c(t) = 0 \Rightarrow (s + 2)Ce^{st} = 0 \Rightarrow s = -2.$$

The natural response is then $y_c(t) = Ce^{-2t}$, where C is yet to be determined.

Because the forcing function is constant, and since the derivative of a constant is zero, the forced response in Step 2 is assumed to be

$$y_p(t) = P,$$

where P is an unknown constant. Substitution of the forced response $y_p(t)$ into the differential equation yields

$$\frac{dP}{dt} + 2P = 0 + 2P = 2,$$

or $y_p(t) = P = 1$. From (3.51), the general solution is

$$y(t) = y_c(t) + y_p(t) = Ce^{-2t} + 1.$$

We now evaluate the coefficient C. The initial condition is given as $y(0) = 4$. The general solution $y(t)$ evaluated at $t = 0$ yields

$$y(0) = y_c(0) + y_p(0) = [Ce^{-2t} + 1]\Big|_{t=0}$$

$$= C + 1 = 4 \Rightarrow C = 3.$$

The total solution is then

$$y(t) = 1 + 3e^{-2t}.$$

General Case

Consider the natural response for the nth-order system

[eq(3.50)] $$\sum_{k=0}^{n} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{m} b_k \frac{d^k x(t)}{dt^k}.$$

The homogeneous equation is formed from (3.50) with the right side set to zero. That is,

$$a_n \frac{d^n y(t)}{dt^n} + a_{n-1} \frac{d^{n-1} y(t)}{dt^{n-1}} + \cdots + a_1 \frac{dy(t)}{dt} + a_0 y(t) = 0, \qquad (3.52)$$

with $a_n \neq 0$. The natural response $y_c(t)$ must satisfy this equation.

We assume that the solution of the homogeneous equation is of the form $y_c(t) = Ce^{st}$. Note that, in (3.52),

$$
\begin{aligned}
y_c(t) &= Ce^{st};\\
\frac{dy_c(t)}{dt} &= Cse^{st};\\
\frac{d^2y_c(t)}{dt^2} &= Cs^2e^{st};\\
&\vdots;\\
\frac{d^ny_c(t)}{dt^n} &= Cs^ne^{st}.
\end{aligned}
\tag{3.53}
$$

Substitution of these terms into (3.57) yields

$$(a_ns^n + a_{n-1}s^{n-1} + \cdots + a_1s + a_0)Ce^{st} = 0. \tag{3.54}$$

If we assume that our solution $y_c(t) = Ce^{st}$ is nontrivial ($C \neq 0$), then, from (3.54),

$$a_ns^n + a_{n-1}s^{n-1} + \cdots + a_1s + a_0 = 0. \tag{3.55}$$

This equation is called the *characteristic equation,* or the *auxiliary equation,* for the differential equation (3.50). The polynomial may be factored as

$$
\begin{aligned}
&a_ns^n + a_{n-1}s^{n-1} + \cdots + a_1s + a_0\\
&\qquad = a_n(s - s_1)(s - s_2)\cdots(s - s_n) = 0.
\end{aligned}
\tag{3.56}
$$

Hence, n values of s, denoted as $s_i, 1 \leqq i \leqq n$, satisfy the equation; that is, $y_{ci}(t) = C_ie^{s_it}$ for the n values of s_i in (3.56) satisfies the homogeneous equation (3.52). Since the differential equation is linear, the sum of these n solutions is also a solution. For the case of no repeated roots, the solution of the homogeneous equation (3.52) may be expressed as

$$y_c(t) = C_1e^{s_1t} + C_2e^{s_2t} + \cdots + C_ne^{s_nt}. \tag{3.57}$$

See Appendix E for the case that the characteristic equation has repeated roots.

3.7 SYSTEM RESPONSE FOR COMPLEX-EXPONENTIAL INPUTS

First in this section, we consider further the linearity property for systems. Then the response of continuous-time LTI systems to a certain class of input signals is derived.

Linearity

Consider the continuous-time LTI system depicted in Figure 3.20. This system is denoted by

$$x(t) \rightarrow y(t). \tag{3.60}$$

For an LTI system, (3.60) can be expressed as the convolution integral

$$y(t) = \int_{-\infty}^{\infty} x(t - \tau)h(\tau)d\tau. \tag{3.61}$$

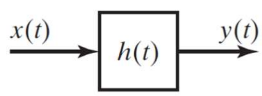


Figure 3.20 LTI system.

The response of an LTI system to the complex-exponential input

$$x(t) = Xe^{st} \tag{3.65}$$

is now investigated. For the general case, both X and s are complex. We investigate the important case that the system of (3.61) and Figure 3.20 is stable and is modeled by an nth-order linear differential equation with constant coefficients. The exponential input of (3.65) is assumed to exist for all time; hence, the *steady-state system response* will be found. In other words, we will find the forced response for a differential equation with constant coefficients for a complex-exponential input signal.

The differential-equation model for an nth-order LTI system is

[eq(3.50)]
$$\sum_{k=0}^{n} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{m} b_k \frac{d^k x(t)}{dt^k},$$

where all a_i and b_i are real constants and $a_n \neq 0$. For the complex-exponential excitation of (3.65), recall from Section 3.5 that the forced response (steady-state response) of (3.50) is of the same mathematical form; hence,

$$y_{ss}(t) = Ye^{st}, \tag{3.66}$$

where $y_{ss}(t)$ is the steady-state response and Y is a complex constant to be determined [s is known from (3.65)]. We denote the forced response as $y_{ss}(t)$ rather than $y_p(t)$, for clarity. From (3.65) and (3.66), the terms of (3.50) become

$$\begin{aligned} a_o y_{ss}(t) &= a_0 Y e^{st} & b_0 x(t) &= b_0 X e^{st} \\ a_1 \frac{dy_{ss}(t)}{dt} &= a_1 s Y e^{st} & b_1 \frac{dx(t)}{dt} &= b_1 s X e^{st} \\ a_2 \frac{d^2 y_{ss}(t)}{dt^2} &= a_2 s^2 Y e^{st} & b_2 \frac{d^2 x(t)}{dt^2} &= b_2 s^2 X e^{st} \\ &\vdots & &\vdots \\ a_n \frac{d^n y_{ss}(t)}{dt^n} &= a_n s^n Y e^{st} & b_m \frac{d^m x(t)}{dt^m} &= b_m s^m X e^{st} \end{aligned} \tag{3.67}$$

These terms are substituted into (3.50), resulting in the equation

$$\begin{aligned} &(a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0) Y e^{st} \\ &\quad = (b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0) X e^{st}. \end{aligned} \tag{3.68}$$

The only unknown in the steady-state response $y_{ss}(t)$ of (3.66) is Y. In (3.68), the factor e^{st} cancels, and Y is given by

$$Y = \left[\frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0} \right] X = H(s)\mathrm{X}. \tag{3.69}$$

It is standard practice to denote the ratio of polynomials as

$$H(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}. \tag{3.70}$$

We show subsequently that this function is related to the impulse response $h(t)$. The function $H(s)$ is called a *transfer function* and is said to be nth order. The order of a transfer function is the same as that of the differential equation upon which the transfer function is based.

We now summarize this development. Consider an LTI system with the transfer function $H(s)$, as given in (3.69) and (3.70). If the system excitation is the complex exponential $X e^{s_1 t}$, the steady-state response is given by, from (3.66) and (3.69),

$$x(t) = X e^{s_1 t} \rightarrow y_{ss}(t) = X H(s_1) e^{s_1 t}. \tag{3.71}$$

The complex-exponential solution in (3.71) also applies for the special case of sinusoidal inputs. Suppose that, in (3.71), $X = |X|e^{j\phi}$ and $s_1 = j\omega_1$, where ϕ and ω_1 are real. Then

$$\begin{aligned} x(t) &= Xe^{s_1 t} = |X|e^{j\phi}e^{j\omega_1 t} = |X|e^{j(\omega_1 t + \phi)} \\ &= |X|\cos(\omega_1 t + \phi) + j|X|\sin(\omega_1 t + \phi). \end{aligned} \tag{3.72}$$

Since, in general, $H(j\omega_1)$ is also complex, we let $H(j\omega_1) = |H(j\omega_1)|e^{j\theta_H}$. The right side of (3.71) can be expressed as

$$\begin{aligned} y_{ss}(t) &= XH(j\omega_1)e^{j\omega_1 t} = |X||H(j\omega_1)|e^{j(\omega_1 t + \phi + \theta_H)} \\ &= |X||H(j\omega_1)|[\cos[\omega_1 t + \phi + \angle H(j\omega_1)] + j\sin[\omega_1 t + \phi + \angle H(j\omega_1)]], \end{aligned}$$

with $\theta_H = \angle H(j\omega_1)$. From (3.64), since the real part of the input signal produces the real part of the output signal,

$$|X|\cos(\omega_1 t + \phi) \rightarrow |X||H(j\omega_1)|\cos[\omega_1 t + \phi + \angle H(j\omega_1)]. \tag{3.73}$$

This result is general for an LTI system and is fundamental to the analysis of LTI systems with periodic inputs; its importance cannot be overemphasized.

EXAMPLE 3.16 **Transfer function of a servomotor**

In this example, we illustrate the transfer function by using a physical device. The device is a servomotor, which is a dc motor used in position control systems. An example of a physical position-control system is the system that controls the position of the read/write heads on a computer hard disk. In addition, the audio compact-disk (CD) player has three position-control systems. (See Section 1.3.)

The input signal to a servomotor is the armature voltage $e(t)$, and the output signal is the motor-shaft angle $\theta(t)$. The commonly used transfer function of a servomotor is second order and is given by [3]

$$H(s) = \frac{K}{s^2 + as},$$

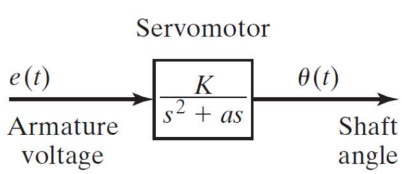


Figure 3.22 System for Example 3.16. ■

where K and a are motor parameters and are determined by the design of the motor. This motor can be represented by the block diagram of Figure 3.22, and the motor differential equation is

$$\frac{d^2\theta(t)}{dt^2} + a\frac{d\theta(t)}{dt} = Ke(t).$$

EXAMPLE 3.17 **Sinusoidal response of an LTI system**

In this example, we calculate the system response of an LTI system with a sinusoidal excitation. Consider a system described by the second-order differential equation

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = 10x(t).$$

Hence, from (3.70), the transfer function is given by

$$H(s) = \frac{10}{s^2 + 3s + 2}.$$

Suppose that the system is excited by the sinusoidal signal $x(t) = 5\cos(2t + 40°)$. In (3.73),

$$H(s)\Big|_{s=j2} X = \frac{10}{s^2 + 3s + 2}\Big|_{s=j2} (5\angle 40°) = \frac{50\angle 40°}{-4 + j6 + 2}$$
$$= \frac{50\angle 40°}{-2 + j6} = \frac{50\angle 40°}{6.325\angle 108.4°} = 7.905e^{-j68.4°}.$$

Thus, from (3.73), the system response is given by

$$y_{ss}(t) = 7.905\cos(2t - 68.4°).$$

Note the calculation required:

$$H(j2) = \frac{10}{(j2)^2 + 3(j2) + 2} = 1.581\angle -108.4°.$$

From (3.73), the steady-state response can be written directly from this numerical value for the transfer function:

$$y_{ss}(t) = (1.581)(5)\cos(2t + 40° - 108.4°)$$
$$= 7.905\cos(2t - 68.4°).$$ ■

Consider now the case in which the input function is a sum of complex exponentials:

$$x(t) = \sum_{k=1}^{N} X_k e^{s_k t}. \tag{3.74}$$

By superposition, from (3.71), the response of an LTI system with the transfer function $H(s)$ is given by

$$y_{ss}(t) = \sum_{k=1}^{N} X_k H(s_k) e^{s_k t}. \tag{3.75}$$

EXAMPLE 3.18 **Transfer function used to calculate LTI system response**

Suppose that the input to the stable LTI system in Figure 3.21, with the transfer function $H(s)$, is given by

$$x(t) = 8 - 5e^{-6t} + 3\cos(4t + 30°).$$

In terms of a complex-exponential input Xe^{st}, the first term in the sum is constant ($s = 0$), the second term is a real exponential ($s = -6$), and the third term is the real part of a complex exponential with $s = j4$, from (3.72). From (3.75), the steady-state response is given by

$$y_{ss}(t) = 8H(0) - 5H(-6)e^{-6t} + 3\mathrm{Re}[H(j4)e^{j(4t+30°)}].$$

The sinusoidal-response term can also be simplified somewhat from (3.73), with the resulting output given by

$$y_{ss}(t) = 8H(0) - 5H(-6)e^{-6t} + 3|H(j4)|\cos[4t + 30° + \angle H(j4)].$$

Impulse Response

Recall that when the impulse response of an LTI system was introduced, the notation $h(\cdot)$ was reserved for the impulse response. In (3.71), the notation $H(\cdot)$ is used to describe the transfer function of an LTI system. It will now be shown that the transfer function $H(s)$ is directly related to the impulse response $h(t)$, and $H(s)$ can be calculated directly from $h(t)$.

For the excitation $x(t) = e^{st}$, the convolution integral (3.15) yields the system response:

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t - \tau)\,d\tau = \int_{-\infty}^{\infty} h(\tau)e^{s(t-\tau)}\,d\tau$$

$$= e^{st}\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}\,d\tau. \tag{3.76}$$

In (3.71), the value of s_1 is not constrained and can be considered to be the variable s. From (3.71) and (3.76),

$$y(t) = e^{st} \int_{-\infty}^{\infty} h(\tau) e^{-s\tau}\, d\tau = H(s) e^{st},$$

and we see that the impulse response and the transfer function of a continuous-time LTI system are related by

$$H(s) = \int_{-\infty}^{\infty} h(t) e^{-st}\, dt. \qquad (3.77)$$

This equation is the desired result. Table 3.1 summarizes the results developed in this section.

We can express these developments in system notation:

$$e^{st} \rightarrow H(s) e^{st}. \qquad (3.78)$$

We see that a complex exponential input signal produces a complex exponential output signal.

***

III. FOURIER SERIES

To introduce the Fourier series, we consider the sum

$$x(t) = 10 + 3\cos\omega_0 t + 5\cos(2\omega_0 t + 30^\circ) + 4\sin 3\omega_0 t. \quad (4.9)$$

This signal is easily shown to be periodic with period $T_0 = 2\pi/\omega_0$. We now manipulate this signal into a different mathematical form, using Euler's relation from Appendix D:

$$x(t) = 10 + \frac{3}{2}[e^{j\omega_0 t} + e^{-j\omega_0 t}] + \frac{5}{2}[e^{j(2\omega_0 t + 30^\circ)} + e^{-j(2\omega_0 t + 30^\circ)}] + \frac{4}{2j}[e^{j3\omega_0 t} - e^{-j3\omega_0 t}],$$

or

$$x(t) = (2e^{j\pi/2})e^{-j3\omega_0 t} + (2.5e^{-j\pi/6})e^{-j2\omega_0 t} + 1.5e^{-j\omega_0 t} + 10 + 1.5e^{j\omega_0 t} + (2.5e^{j\pi/6})e^{j2\omega_0 t} + (2e^{-j\pi/2})e^{j3\omega_0 t}. \quad (4.10)$$

This equation can be expressed in the compact form

$$x(t) = C_{-3}e^{-j3\omega_0 t} + C_{-2}e^{-j2\omega_0 t} + C_{-1}e^{-j\omega_0 t} + C_0 + C_1 e^{j\omega_0 t} + C_2 e^{j2\omega_0 t} + C_3 e^{j3\omega_0 t} = \sum_{k=-3}^{3} C_k e^{jk\omega_0 t}.$$

The coefficients C_k for this series of complex exponential functions are listed in Table 4.1. Note that $C_k = C_{-k}^*$, where the asterisk indicates the complex conjugate.

We see then that a sum of sinusoidal functions can be converted to a sum of complex exponential functions. Note that even though some of the terms are complex, the sum is real. As shown next, (4.10) is one form of the Fourier series.

TABLE 4.1 Coefficients for Example 4.2

K	C_k	C_{-k}
0	10	—
1	1.5	1.5
2	$2.5\angle 30^\circ$	$2.5\angle -30^\circ$
3	$2\angle -90^\circ$	$2\angle 90^\circ$

Given a real periodic signal $x(t)$, a *harmonic series* for this signal is defined as

$$x(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t}; \quad C_k = C_{-k}^*. \tag{4.11}$$

The frequency ω_0 is called the *fundamental frequency* or the *first harmonic,* and the frequency $k\omega_0$ is called the kth *harmonic*. If the coefficients C_k and the signal $x(t)$ in (4.11) are related by an equation to be developed later, this harmonic series is a *Fourier series*. For this case, the summation (4.11) is called the *complex exponential form,* or simply the exponential form, of the Fourier series; the coefficients C_k are called the *Fourier coefficients*. Equation (4.10) is an example of a Fourier series in the exponential form. We next derive a second form of the Fourier series.

The general coefficient C_k in (4.11) is complex, as indicated in Table 4.1, with C_{-k} equal to the conjugate of C_k. The coefficient C_k can be expressed as

$$C_k = |C_k|e^{j\theta_k},$$

with $-\infty < k < \infty$. Since $C_{-k} = C_k^*$, it follows that $\theta_{-k} = -\theta_k$. For a given value of k, the sum of the two terms of the same frequency $k\omega_0$ in (4.11) yields

$$\begin{aligned} C_{-k}e^{-jk\omega_0 t} + C_k e^{jk\omega_0 t} &= |C_k|e^{-j\theta_k}e^{-jk\omega_0 t} + |C_k|e^{j\theta_k}e^{jk\omega_0 t} \\ &= |C_k|[e^{-j(k\omega_0 t+\theta_k)} + e^{j(k\omega_0 t+\theta_k)}] \\ &= 2|C_k|\cos(k\omega_0 t + \theta_k). \end{aligned} \tag{4.12}$$

Hence, given the Fourier coefficients C_k, we can easily find the *combined trigonometric form* of the Fourier series:

$$x(t) = C_0 + \sum_{k=1}^{\infty} 2|C_k|\cos(k\omega_0 t + \theta_k). \tag{4.13}$$

A third form of the Fourier series can be derived from (4.13). From Appendix A, we have the trigonometric identity

$$\cos(a + b) = \cos a \cos b - \sin a \sin b. \quad (4.14)$$

The use of this identity with (4.13) yields

$$x(t) = C_0 + \sum_{k=1}^{\infty} 2|C_k|\cos(k\omega_0 t + \theta_k)$$

$$= C_0 + \sum_{k=1}^{\infty} [2|C_k|\cos\theta_k \cos k\omega_0 t - 2|C_k| \sin\theta_k \sin k\omega_0 t]. \quad (4.15)$$

From Euler's relationship, we define the coefficients A_k and B_k implicitly via the formula

$$2C_k = 2|C_k|e^{j\theta_k}$$

$$= 2|C_k|\cos\theta_k + j2|C_k| \sin\theta_k = A_k - jB_k, \quad (4.16)$$

where A_k and B_k are real. Substituting (4.16) into (4.15) yields the *trigonometric form* of the Fourier series

$$x(t) = A_0 + \sum_{k=1}^{\infty} [A_k \cos k\omega_0 t + B_k \sin k\omega_0 t], \quad (4.17)$$

with $A_0 = C_0$. The original work of Joseph Fourier (1768–1830) involved the series in this form.

The three forms of the Fourier series [(4.11), (4.13), (4.17)] are listed in Table 4.2. Also given is the equation for calculating the coefficients; this equation is developed later. From (4.16), the coefficients of the three forms are related by

$$2C_k = A_k - jB_k; \quad C_k = |C_k|e^{j\theta_k}; \quad C_0 = A_0. \quad (4.18)$$

Recall that A_k and B_k are real and, in general, C_k is complex.

[eq(4.11)] $$x(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t},$$

TABLE 4.2 Forms of the Fourier Series

Name	Equation
Exponential	$\sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t}; \quad C_k = \lvert C_k \rvert e^{j\theta_k}, C_{-k} = C_k^*$
Combined trigonometric	$C_0 + \sum_{k=1}^{\infty} 2\lvert C_k \rvert \cos(k\omega_0 t + \theta_k)$
Trigonometric	$A_0 + \sum_{k=1}^{\infty} (A_k \cos k\omega_0 t + B_k \sin k\omega_0 t)$
	$2C_k = A_k - jB_k, C_0 = A_0$
Coefficients	$C_k = \frac{1}{T_0} \int_{T_0} x(t) e^{-jk\omega_0 t} dt$

$$C_k = \frac{1}{T_0} \int_{T_0} x(t) e^{-jk\omega_0 t}\, dt. \tag{4.23}$$

We now consider the coefficient C_0. From (4.23),

$$C_0 = \frac{1}{T_0} \int_{T_0} x(t)\, dt.$$

Hence, C_0 is the *average value* of the signal $x(t)$. This average value is also called the *dc value,* a term that originated in circuit analysis. For some waveforms, the dc value can be found by inspection.

EXAMPLE 4.2 **Fourier series of a square wave**

Consider the square wave of Figure 4.4. This signal is common in physical systems. For example, this signal appears in many electronic oscillators as an intermediate step in the generation of a sinusoid.

We now calculate the Fourier coefficients of the square wave. Because

$$x(t) = \begin{cases} V, & 0 < t < T_0/2 \\ -V, & T_0/2 < t < T_0 \end{cases},$$

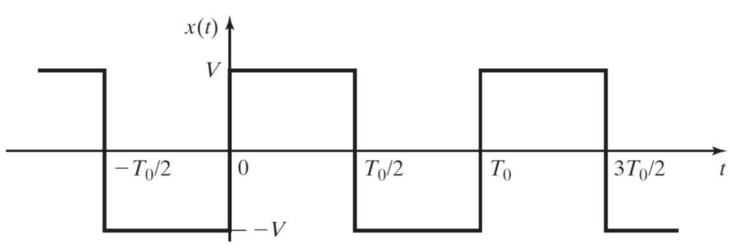


Figure 4.4 Square wave with amplitude V.

$$\begin{aligned}C_k &= \frac{1}{T_0}\int_{T_0} x(t)e^{-jk\omega_0 t}\,dt \\ &= \frac{V}{T_0}\int_0^{T_0/2} e^{-jk\omega_0 t}\,dt - \frac{V}{T_0}\int_{T_0/2}^{T_0} e^{-jk\omega_0 t}\,dt \\ &= \frac{V}{T_0(-jk\omega_0)}\left[e^{-jk\omega_0 t}\Big|_0^{T_0/2} - e^{-jk\omega_0 t}\Big|_{T_0/2}^{T_0}\right].\end{aligned}$$

$$C_k = \frac{jV}{2\pi k}(e^{-jk\pi} - e^{-j0} - e^{-jk2\pi} + e^{-jk\pi})$$

$$= \begin{cases} -\dfrac{2jV}{k\pi} = \dfrac{2V}{k\pi}\angle -90^\circ, & k \text{ odd} \\ 0, & k \text{ even} \end{cases}, \tag{4.24}$$

with $C_0 = 0$. The value of C_0 is seen by inspection, since the square wave has an average value of zero. Also, C_0 can be calculated from (4.24) by L'Hôpital's rule, Appendix B.

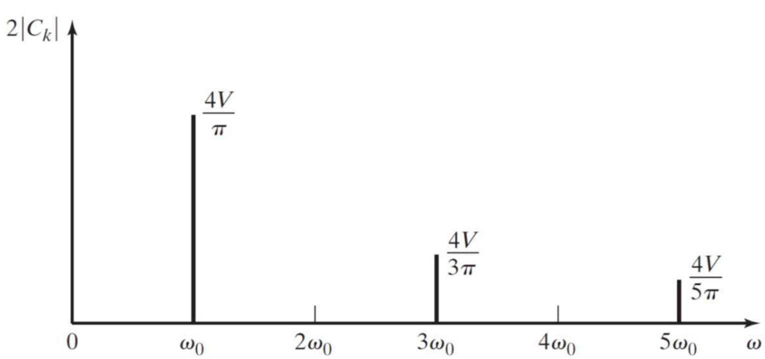
The exponential form of the Fourier series of the square wave is then

$$x(t) = \sum_{\substack{k=-\infty \\ k \text{ odd}}}^{\infty} \frac{2V}{k\pi} e^{-j\pi/2} e^{jk\omega_0 t}. \tag{4.25}$$

The combined trigonometric form is given by

$$x(t) = \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{4V}{k\pi}\cos(k\omega_0 t - 90^\circ) \tag{4.26}$$

Frequency Spectra



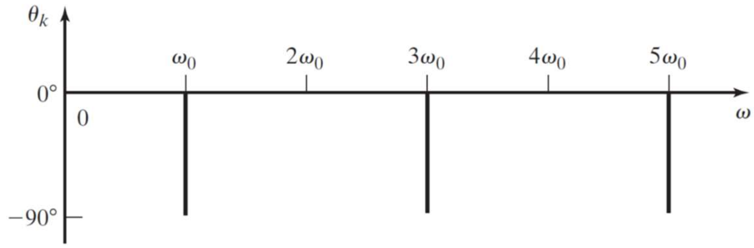


Figure 4.5 Frequency spectrum for a square wave.

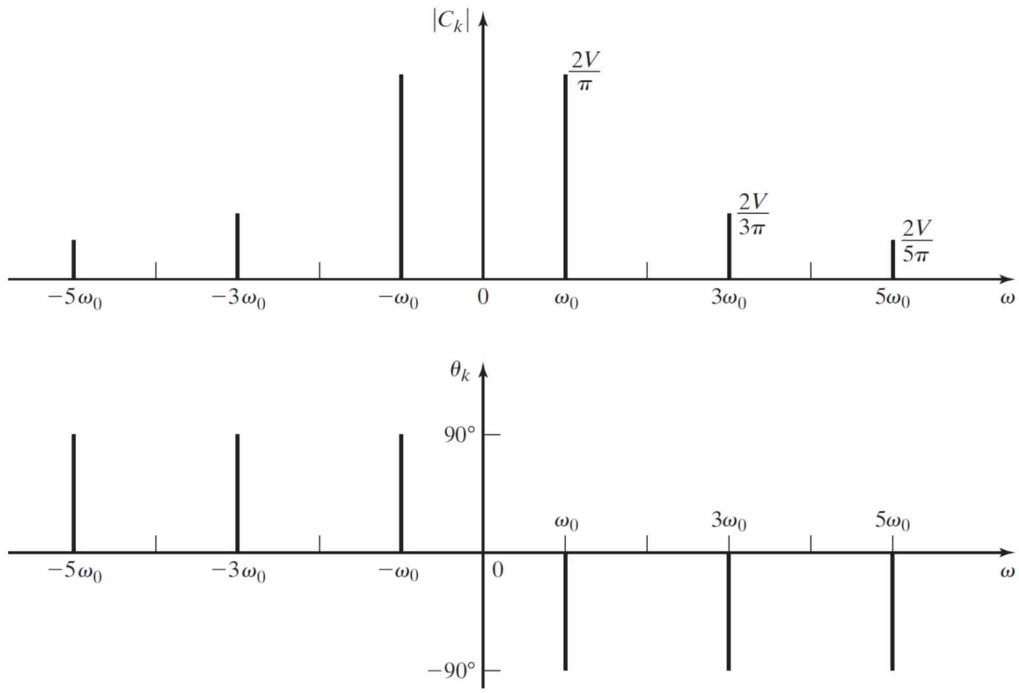


Figure 4.6 Frequency spectrum for a square wave.

EXAMPLE 4.5 **Fourier series for an impulse train**

The Fourier series for the impulse train shown in Figure 4.10 will be calculated. From (4.23),

$$C_k = \frac{1}{T_0}\int_{T_0} x(t)e^{-jk\omega_0 t}\,dt$$

$$= \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \delta(t)e^{-jk\omega_0 t}\,dt = \frac{1}{T_0}e^{-jk\omega_0 t}\Big|_{t=0} = \frac{1}{T_0}.$$

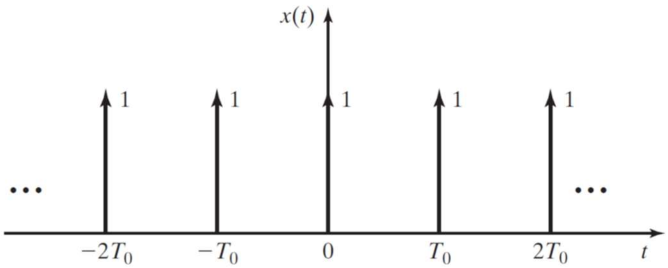


Figure 4.10 Impulse train.

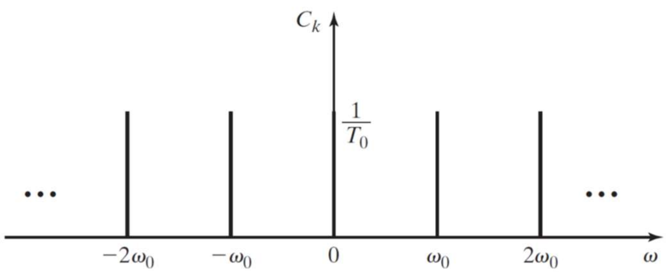


Figure 4.11 Frequency spectrum for an impulse train.

$$x(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T_0} e^{jk\omega_0 t}. \tag{4.27}$$

A line spectrum for this function is given in Figure 4.11. Because the Fourier coefficients are real, no phase plot is given. From (4.13), the combined trigonometric form for the train of impulse functions is given by

$$x(t) = \frac{1}{T_0} + \sum_{k=1}^{\infty} \frac{2}{T_0} \cos k\omega_0 t.$$

■

EXAMPLE 4.6 **Frequency spectrum of a rectangular pulse train**

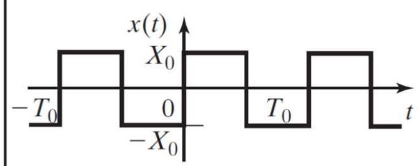
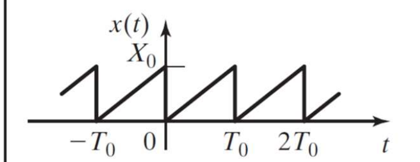
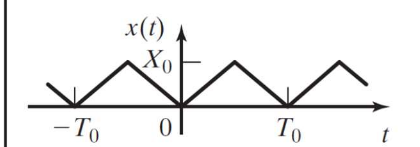
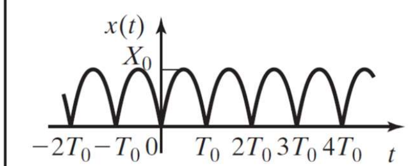
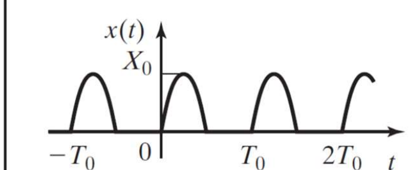
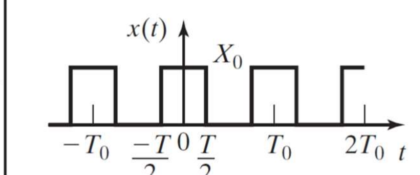
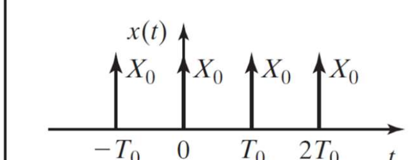
For this example, the frequency spectrum of the rectangular pulse train of Figure 4.12 will be plotted. This waveform is common in engineering. The clock signal in a digital computer is a rectangular pulse train of this form. Also, in communications, one method of modulation is to vary the amplitudes of the rectangular pulses in a pulse train according to the information to be transmitted. This method of modulation, called pulse-amplitude modulation, is described in Sections 1.3 and 6.6.

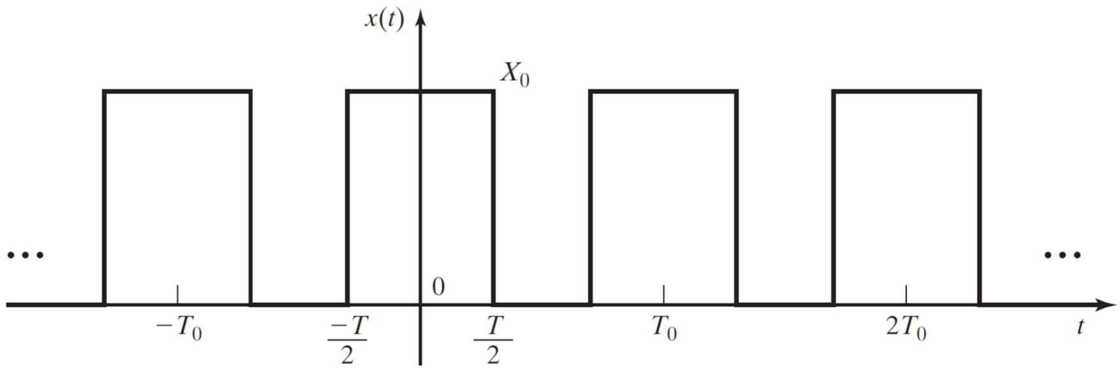
From Table 4.3, the Fourier series for this signal is given by

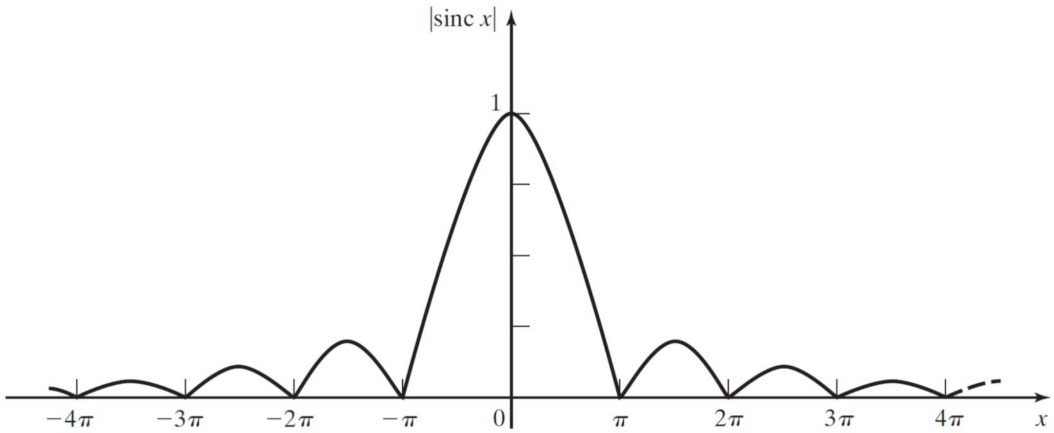
$$x(t) = \sum_{k=-\infty}^{\infty} \frac{TX_0}{T_0} \operatorname{sinc} \frac{Tk\omega_0}{2} e^{jk\omega_0 t}, \tag{4.28}$$

$$\operatorname{sinc} x = \frac{\sin x}{x}. \tag{4.29}$$

TABLE 4.3 Fourier Series for Common Signals

Name	Waveform	C_0	$C_k, k \neq 0$	Comments
1. Square wave		0	$-j\dfrac{2X_0}{\pi k}$	$C_k = 0$, k even
2. Sawtooth		$\dfrac{X_0}{2}$	$j\dfrac{X_0}{2\pi k}$	
3. Triangular wave		$\dfrac{X_0}{2}$	$\dfrac{-2X_0}{(\pi k)^2}$	$C_k = 0$, k even
4. Full-wave rectified		$\dfrac{2X_0}{\pi}$	$\dfrac{-2X_0}{\pi(4k^2-1)}$	
5. Half-wave rectified		$\dfrac{X_0}{\pi}$	$\dfrac{-X_0}{\pi(k^2-1)}$	$C_k = 0$, k odd, except $C_1 = -j\dfrac{X_0}{4}$ and $C_{-1} = j\dfrac{X_0}{4}$
6. Rectangular wave		$\dfrac{TX_0}{T_0}$	$\dfrac{TX_0}{T_0}\,\text{sinc}\,\dfrac{Tk\omega_0}{2}$	$\dfrac{Tk\omega_0}{2} = \dfrac{\pi Tk}{T_0}$
7. Impulse train		$\dfrac{X_0}{T_0}$	$\dfrac{X_0}{T_0}$	





EXAMPLE 4.7 **LTI system response for a square-wave input**

Suppose that for the LTI system of Figure 4.16, the impulse response and the transfer function are given by

$$h(t) = e^{-t}u(t) \Leftrightarrow H(s) = \frac{1}{s+1}.$$

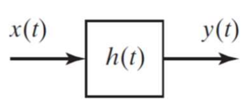


Figure 4.16 LTI system.

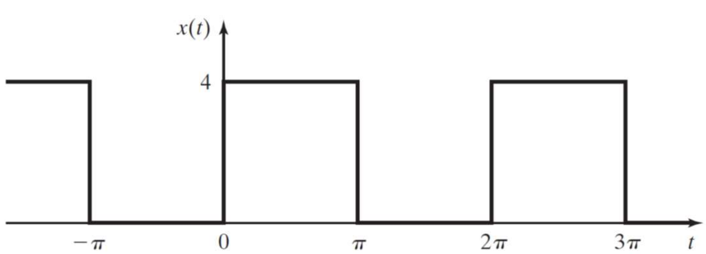


Figure 4.18 Input signal for Example 4.7.

$$x(t) = C_{0x} + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} C_{kx} e^{jk\omega_0 t} = 2 + \sum_{\substack{k=-\infty \\ k \text{ odd}}}^{\infty} \frac{4}{\pi k} e^{-j\pi/2} e^{jkt}.$$

$$H(jk\omega_0)|_{\omega_0=1} = \frac{1}{1+jk} = \frac{1}{\sqrt{1+k^2}} \angle \tan^{-1}(-k).$$

For k odd, from (4.39),

$$C_{ky} = H(jk\omega_0) C_{kx} = \frac{1}{\sqrt{1+k^2}} \left[\frac{4}{\pi k}\right] \angle -\pi/2 - \tan^{-1}(k)$$

and

$$C_{0y} = H(j0) C_{0x} = (1)(2) = 2.$$

TABLE 4.6 Fourier Coefficients for Example 4.7

k	$H(jk\omega_0)$	C_{kx}	C_{ky}	$\lvert C_{kx}\rvert$	$\lvert C_{ky}\rvert$
0	1	2	2	2	2
1	$\frac{1}{\sqrt{2}}\angle -45^\circ$	$\frac{4}{\pi}\angle -90^\circ$	$\frac{4}{\pi\sqrt{2}}\angle -135^\circ$	1.273	0.900
3	$\frac{1}{\sqrt{10}}\angle -71.6^\circ$	$\frac{4}{3\pi}\angle -90^\circ$	$\frac{4}{3\pi\sqrt{10}}\angle -161.6^\circ$	0.424	0.134
5	$\frac{1}{\sqrt{26}}\angle -78.7^\circ$	$\frac{4}{5\pi}\angle -90^\circ$	$\frac{4}{5\pi\sqrt{26}}\angle -168.7^\circ$	0.255	0.050

***

IV. Fourier Transform

Sufficient conditions for the existence of the Fourier transform are similar to those given earlier for the Fourier series. They are the *Dirichlet conditions*:

1. On any finite interval,
   - **a.** $f(t)$ is bounded;
   - **b.** $f(t)$ has a finite number of maxima and minima; and
   - **c.** $f(t)$ has a finite number of discontinuities.
2. $f(t)$ is absolutely integrable; that is,

$$\int_{-\infty}^{\infty} |f(t)|\, dt < \infty.$$

$$\mathcal{F}\{f(t)\} = F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\, dt. \tag{5.1}$$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}\, d\omega = F^{-1}\{F(\omega)\}, \tag{5.2}$$

Together, these equations are called a *transform pair*, and their relationship is often represented in mathematical notation as

$$f(t) \overset{\mathcal{F}}{\longleftrightarrow} F(\omega).$$

The continuous frequency spectrum shown in Figure 5.3 is a graphical representation of the Fourier transform of a single rectangular pulse of amplitude V and duration T (which can also be considered to be a periodic pulse of infinite period). The analytical expression for the Fourier transform is found by (5.1). The rectangular pulse can be described mathematically as the sum of two step functions:

$$f(t) = Vu(t + T/2) - Vu(t - T/2).$$

To simplify the integration in (5.1), we can recognize that $f(t)$ has a value of V during the period $-T/2 < t < +T/2$ and is zero for all other times. Then,

$$F(\omega) = \int_{-T/2}^{+T/2} Ve^{-j\omega t}\, dt = V\left[\frac{e^{-j\omega t}}{-j\omega}\Bigg|_{-T/2}^{+T/2}\right]$$

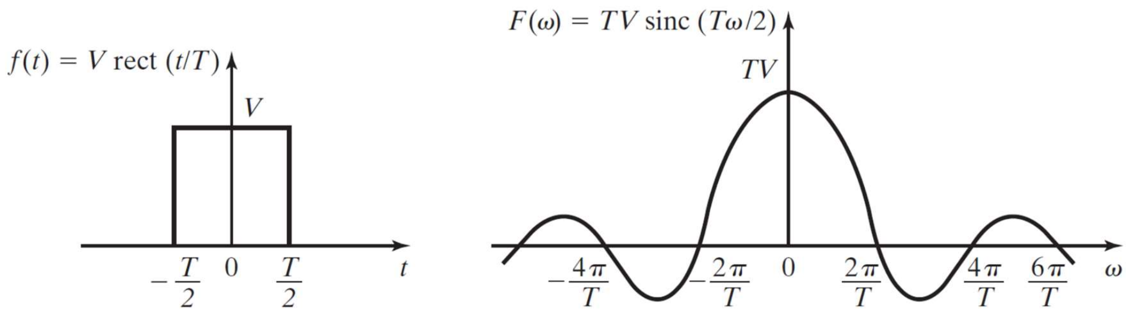


Figure 5.3 A rectangular pulse and its Fourier transform.

$$= V\left[\frac{e^{-jT\omega/2} - e^{+jT\omega/2}}{-j\omega}\right] = \frac{TV}{\omega T/2}\left[\frac{e^{jT\omega/2} - e^{-jT\omega/2}}{j2}\right]$$

$$= TV\left[\frac{\sin(T\omega/2)}{T\omega/2}\right] = TV\ \text{sinc}(T\omega/2),$$

and we have derived our first Fourier transform:

$$\mathscr{F}\{V[u(t + T/2) - u(t - T/2)]\} = TV\ \text{sinc}(T\omega/2).$$

$$\text{rect}(t/T) = [u(t + T/2) - u(t - T/2)].$$

Therefore, in our table of transform pairs we will list

$$\text{rect}(t/T) \overset{\mathscr{F}}{\longleftrightarrow} T\ \text{sinc}(T\omega/2) \tag{5.4}$$

Any useful signal $f(t)$ that meets the condition

$$E = \int_{-\infty}^{\infty} |f(t)|^2\, dt < \infty \tag{5.5}$$

is absolutely integrable. In (5.5), E is the energy associated with the signal, which can be seen if we consider the signal $f(t)$ to be the voltage across a 1-Ω resistor. The power delivered by $f(t)$ is then

$$p(t) = |f(t)|^2/R = |f(t)|^2,$$

and the integral of power over time is energy.

A signal that meets the condition of containing finite energy is known as an *energy signal*. Energy signals generally include nonperiodic signals that have a finite time duration (such as the rectangular function, which is considered in several examples) and signals that approach zero asymptotically so that $f(t)$ approaches zero as t approaches infinity.

A signal that meets the condition

$$P = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} |f(t)|^2\,dt < \infty \tag{5.6}$$

is called a *power signal*.

An example of a mathematical function that does not have a Fourier transform, because it does not meet the Dirichlet condition of absolute integrability, is $f(t) = e^{-t}$. However, the frequently encountered signal $f(t) = e^{-t}u(t)$ does meet the Dirichlet conditions and does have a Fourier transform.

The impulse function, in fact, provides a building block for several of the more important transform pairs. Consider the waveform

$$f(t) = A\delta(t - t_0),$$

which represents an impulse function of weight A that is nonzero only at time $t = t_0$, as illustrated in Figure 5.4(a). (See Section 2.4.) The Fourier transform of this waveform is

$$F(\omega) = \mathcal{F}\{A\delta(t - t_0)\} = \int_{-\infty}^{\infty} A\delta(t - t_0)e^{-j\omega t}\,dt.$$

Recall the sifting property of the impulse function described in (2.41), namely, that

$$\int_{-\infty}^{\infty} f(t)\delta(t - t_0)\,dt = f(t_0),$$

for $f(t)$ continuous at $t = t_0$. Using this property of the impulse function to evaluate the Fourier transform integral, we find that

$$\mathcal{F}\{A\delta(t - t_0)\} = Ae^{-j\omega t_0}. \tag{5.7}$$

$$\delta(t) \overset{\mathcal{F}}{\longleftrightarrow} 1. \tag{5.8}$$

While we are dealing with the impulse function, let's consider the case of an impulse function in the frequency domain. We have

$$F(\omega) = \delta(\omega - \omega_0),$$

The inverse Fourier transform of this impulse function is found from Equation (5.2):

$$f(t) = \mathscr{F}^{-1}\{F(\omega)\} = \mathscr{F}^{-1}\{\delta(\omega - \omega_0)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega - \omega_0)e^{j\omega t}d\omega.$$

After applying the sifting property of the impulse function, we have

$$f(t) = \mathscr{F}^{-1}\{\delta(\omega - \omega_0)\} = \frac{1}{2\pi}e^{j\omega_0 t},$$

which is recognized to be a complex phasor of constant magnitude that rotates in phase at a frequency of ω_0 rad/s.

The Fourier transform pair

$$e^{j\omega_0 t} \xleftrightarrow{\mathscr{F}} 2\pi\delta(\omega - \omega_0) \tag{5.9}$$

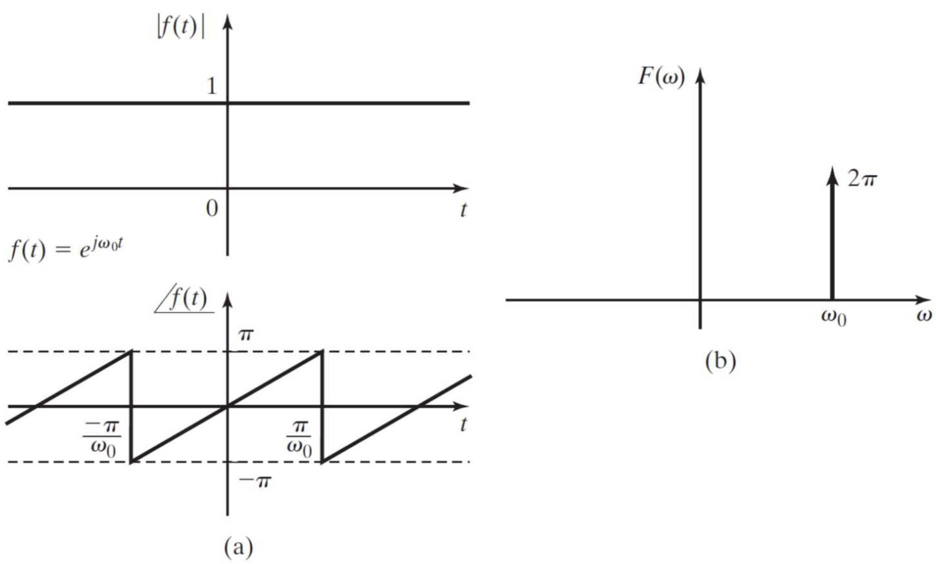


Figure 5.5 Time-domain plots and frequency spectra of $e^{j\omega_0 t}$.

5.2 PROPERTIES OF THE FOURIER TRANSFORM

$$f_1(t) \xleftrightarrow{\mathscr{F}} F_1(\omega) \quad \text{and} \quad f_2(t) \xleftrightarrow{\mathscr{F}} F_2(\omega),$$

Linearity

$$[af_1(t) + bf_2(t)] \xleftrightarrow{\mathscr{F}} [aF_1(\omega) + bF_2(\omega)], \tag{5.10}$$

EXAMPLE 5.2 **The linearity property of the Fourier transform**

We can make use of the property of linearity to find the Fourier transforms of some types of waveforms. For example, consider

$$f(t) = B\cos\omega_0 t.$$

$$f(t) = \frac{B}{2}[e^{j\omega_0 t} + e^{-j\omega_0 t}] = \frac{B}{2}e^{j\omega_0 t} + \frac{B}{2}e^{-j\omega_0 t}.$$

$$B\cos\omega_0 t \xleftrightarrow{\mathscr{F}} \pi B[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)], \tag{5.11}$$

Time Scaling

The time-scaling property provides that if

$$f(t) \xleftrightarrow{\mathscr{F}} F(\omega),$$

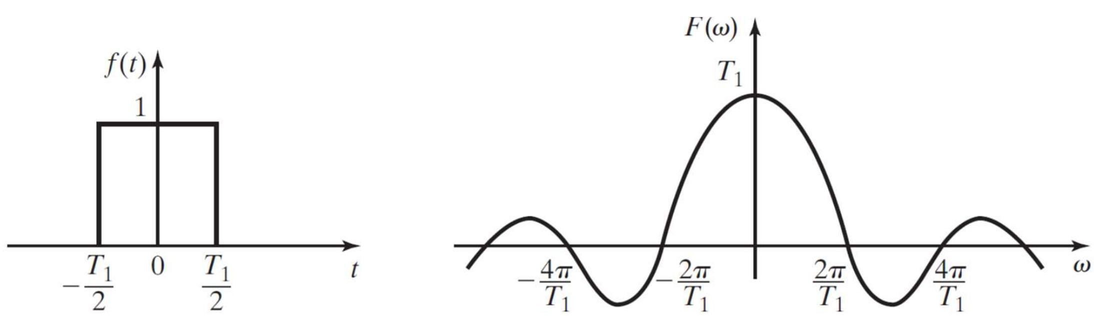
then, for a constant scaling factor a,

$$f(at) \xleftrightarrow{\mathscr{F}} \frac{1}{|a|}F\left(\frac{\omega}{a}\right). \tag{5.12}$$

EXAMPLE 5.3 **The time-scaling property of the Fourier transform**

We now find the Fourier transform of the rectangular waveform

$$g(t) = \text{rect}(2t/T_1).$$



(a)

From the result of Example 5.1,

[eq(5.4)]
$$V \operatorname{rect}(t/T) \overset{\mathcal{F}}{\longleftrightarrow} TV \operatorname{sinc}(T\omega/2),$$

$$g(t) = f(2t),$$

where

$$f(t) = \operatorname{rect}(t/T_1).$$

Therefore, from (5.12),

$$G(\omega) = \frac{1}{2}F(\omega/2) = \frac{T_1}{2}\operatorname{sinc}\left(\frac{\omega T_1}{4}\right).$$

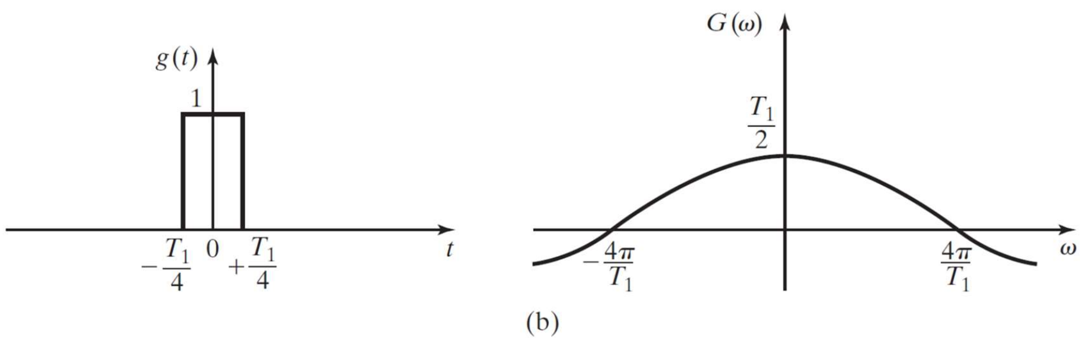


Figure 5.6 Rectangular pulses and their frequency spectra.

Time Shifting

The property of time shifting previously appeared in the Fourier transform of the *impulse function* (5.7) derived in Section 5.1, although it was not recognized at that time. This property is stated mathematically as

$$f(t - t_0) \overset{\mathcal{F}}{\longleftrightarrow} F(\omega)e^{-j\omega t_0}, \tag{5.13}$$

where the symbol t_0 represents the amount of shift in time.

EXAMPLE 5.4 **The time-shifting property of the Fourier transform**

We now find the Fourier transform of the impulse function, which occurs at time zero. From (5.8),

$$\mathcal{F}\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(t)e^{-j\omega t}dt = e^{-j\omega t}\Big|_{t=0} = 1.$$

If the impulse function is shifted in time so that it occurs at time t_0 instead of at $t = 0$, we see from the time-shifting property (5.13) that

$$\mathscr{F}\{\delta(t - t_0)\} = (1)e^{-j\omega t_0} = e^{-j\omega t_0},$$

which is recognized as the same result obtained in (5.7). ■

EXAMPLE 5.5 **Fourier transform of a time-delayed sinusoidal signal**

Consider the time-shifted cosine wave of frequency $\omega = 200\pi$ and a delay of 1.25 ms in its propagation:

$$x(t) = 10\cos[200\pi(t - 1.25 \times 10^{-3})].$$

This signal can be viewed as a phase-shifted cosine wave where the amount of phase shift is $\pi/4$ radians:

$$x(t) = 10\cos(200\pi t - \pi/4).$$

Using the linearity and time-shifting property, we find the Fourier transform of this delayed cosine wave:

$$\begin{aligned}\mathscr{F}\{x(t)\} &= X(\omega) = 10\mathscr{F}\{\cos(200\pi t)\}e^{-j.00125\omega} \\ &= 10\pi[\delta(\omega - 200\pi) + \delta(\omega + 200\pi)]e^{-j.00125\omega} \\ &= 10\pi[\delta(\omega - 200\pi)e^{-j\pi/4} + \delta(\omega + 200\pi)e^{j\pi/4}].\end{aligned}$$

The rotating phasor, $e^{-j.00125\omega}$, is reduced to the two fixed phasors shown in the final equation, because the frequency spectrum has zero magnitude except at $\omega = 200\pi$ and $\omega = -200\pi$. Recall, from Table 2.3, that

$$F(\omega)\delta(\omega - \omega_0) = F(\omega_0)\delta(\omega - \omega_0).$$

Time Transformation

$$f(at) \overset{\mathscr{F}}{\longleftrightarrow} \frac{1}{|a|}F\left(\frac{\omega}{a}\right).$$

Application of the time-shift property (5.13) to this time-scaled function gives us the time-transformation property:

$$f(at - t_0) \overset{\mathscr{F}}{\longleftrightarrow} \frac{1}{|a|}F\left(\frac{\omega}{a}\right)e^{-jt_0(\omega/a)}. \quad (5.14)$$

Duality

The duality property, which is sometimes known as the *symmetry* property, is stated as

$$F(t) \overset{\mathscr{F}}{\longleftrightarrow} 2\pi f(-\omega) \quad \text{when} \quad f(t) \overset{\mathscr{F}}{\longleftrightarrow} F(\omega). \quad (5.15)$$

$$V\,\mathrm{rect}(t/T) \stackrel{\mathscr{F}}{\longleftrightarrow} TV\,\mathrm{sinc}(\omega T/2).$$

$$TV\,\mathrm{sinc}(Tt/2) \stackrel{\mathscr{F}}{\longleftrightarrow} 2\pi V\,\mathrm{rect}(-\omega/T).$$

Because the waveform is an even function of frequency—in other words, $F(-\omega) = F(\omega)$—we can rewrite the equation that describes the waveform as

$$2\pi f(-\omega) = 2\pi A\,\mathrm{rect}(\omega/2\beta),$$

where we have substituted $T = 2\beta$ and $V = A$. The duality property can be used with these values substituted into (5.4) to determine that

$$F(t) = 2\beta\; A\,\mathrm{sinc}(\beta t).$$

The transform pair

$$\frac{A\beta}{\pi}\,\mathrm{sinc}\,(\beta t) \stackrel{\mathscr{F}}{\longleftrightarrow} A\,\mathrm{rect}(\omega/2\beta)$$

is shown in Figures 5.8(b) and (a), respectively. ■

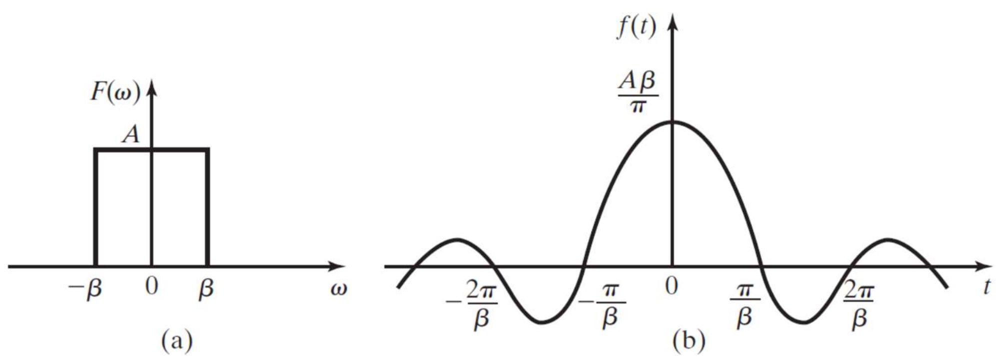


Figure 5.8 A rectangular pulse in the frequency domain.

Convolution

The convolution property states that if

$$f_1(t) \stackrel{\mathscr{F}}{\longleftrightarrow} F_1(\omega) \quad \text{and} \quad f_2 \stackrel{\mathscr{F}}{\longleftrightarrow} F_2(\omega),$$

then convolution of the time-domain waveforms has the effect of multiplying their frequency-domain counterparts. Thus,

$$f_1(t)*f_2(t) \xleftrightarrow{\mathcal{F}} F_1(\omega)F_2(\omega), \tag{5.16}$$

where

$$f_1(t)*f_2(t) = \int_{-\infty}^{\infty} f_1(\tau)f_2(t-\tau)d\tau = \int_{-\infty}^{\infty} f_1(t-\tau)f_2(\tau)d\tau.$$

$$y(t) = x(t)*h(t) \xleftrightarrow{\mathfrak{F}} X(\omega)H(\omega) = Y(\omega)$$

$$f_1(t)f_2(t) \xleftrightarrow{\mathcal{F}} \frac{1}{2\pi}F_1(\omega)*F_2(\omega), \tag{5.17}$$

where

$$F_1(\omega)*F_2(\omega) = \int_{-\infty}^{\infty} F_1(\lambda)F_2(\omega-\lambda)d\lambda = \int_{-\infty}^{\infty} F_1(\omega-\lambda)F_2(\lambda)d\lambda.$$

EXAMPLE 5.8 **The time-convolution property of the Fourier transform**

Chapter 3 discusses the response of linear time-invariant systems to input signals. A block diagram of a linear system is shown in Figure 5.9(a). If the output of the system in response to an impulse function at the input is described as $h(t)$, then $h(t)$ is called the *impulse response* of the system. The output of the system in response to any input signal can then be determined by convolution of the impulse response, $h(t)$, and the input signal, $x(t)$:

$$y(t) = x(t)*h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau.$$

Using the convolution property of the Fourier transform, we can find the frequency spectrum of the output signal from

$$Y(\omega) = X(\omega)H(\omega),$$

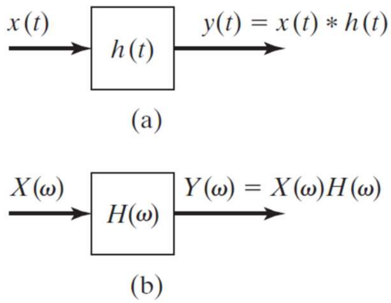


Figure 5.9 A linear time-invariant system.

where

$$h(t) \overset{\mathscr{F}}{\longleftrightarrow} H(\omega),\; x(t) \overset{\mathscr{F}}{\longleftrightarrow} X(\omega), \quad \text{and} \quad y(t) \overset{\mathscr{F}}{\longleftrightarrow} Y(\omega).$$

Frequency Shifting

The frequency shifting property is stated mathematically as

$$x(t)e^{j\omega_0 t} \overset{\mathscr{F}}{\longleftrightarrow} X(\omega - \omega_0). \tag{5.18}$$

This property was demonstrated in the derivation of Equation (5.9), without our having recognized it.

EXAMPLE 5.9 **The frequency-shift property of the Fourier transform**

In the generation of communication signals, often two signals such as

$$g_1(t) = 2\cos(200\pi t) \quad \text{and} \quad g_2(t) = 5\cos(1000\pi t)$$

are multiplied together to give

$$g_3(t) = g_1(t)g_2(t) = 10\cos(200\pi t)\cos(1000\pi t).$$

We can use the frequency-shifting property to find the frequency spectrum of $g_3(t)$. We rewrite the product waveform $g_3(t)$ by using Euler's identity on the second cosine factor:

$$\begin{aligned} g_3(t) &= 10\cos(200\pi t)\frac{e^{j1000\pi t} + e^{-j1000\pi t}}{2} \\ &= 5\cos(200\pi t)e^{j1000\pi t} + 5\cos(200\pi t)e^{-j1000\pi t}. \end{aligned}$$

The Fourier transform of this expression is found from the properties of linearity (5.10), frequency shifting (5.18), and the transform of $\cos(\omega_0 t)$ from (5.11):

$$G_3(\omega) = 5\pi[\delta(\omega - 200\pi - 1000\pi) + \delta(\omega + 200\pi - 1000\pi)] + 5\pi[\delta(\omega - 200\pi + 1000\pi) + \delta(\omega + 200\pi + 1000\pi)].$$

In final form, we write

$$G_3(\omega) = 5\pi[\delta(\omega - 1200\pi) + \delta(\omega - 800\pi) + \delta(\omega + 800\pi) + \delta(\omega + 1200\pi)].$$

The frequency spectra of $g_1(t)$, $g_2(t)$, and $g_3(t)$ are shown in Figure 5.10.

It is of interest to engineers that the inverse Fourier transform of $G_3(\omega)$ is

$$\begin{aligned} g_3(t) &= \mathcal{F}^{-1}\{5\pi[\delta(\omega - 1200\pi) + \delta(\omega + 1200\pi)]\} \\ &\quad + \mathcal{F}^{-1}\{5\pi[\delta(\omega - 800\pi) + \delta(\omega + 800\pi)]\} \\ &= 5\cos 1200\pi t + 5\cos 800\pi t. \end{aligned}$$

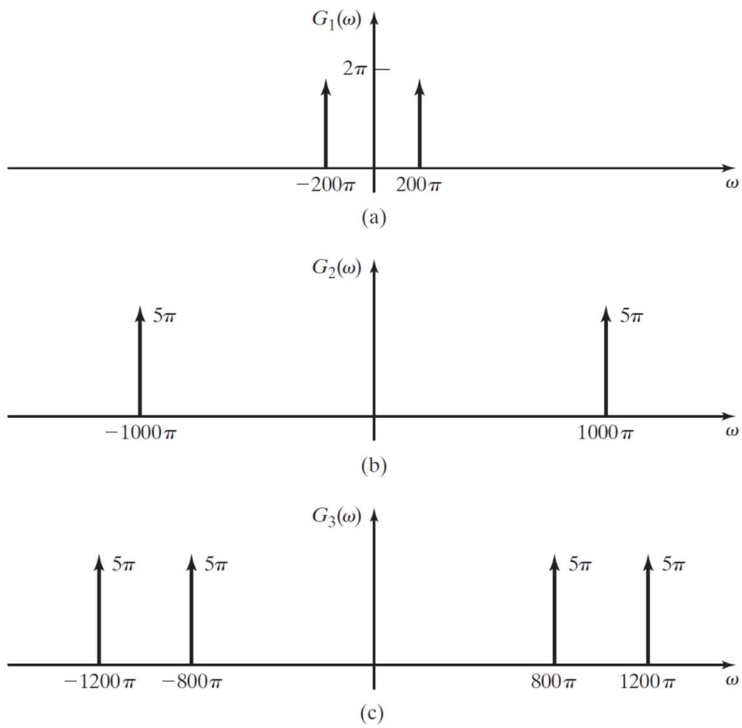


Figure 5.10 The frequency spectrum of $10\cos(200\pi t)\cos(1000\pi t)$.

Time Differentiation

If

$$f(t) \xleftrightarrow{\mathcal{F}} F(\omega),$$

$$\frac{d[f(t)]}{dt} \overset{\mathscr{F}}{\longleftrightarrow} j\omega F(\omega). \tag{5.19}$$

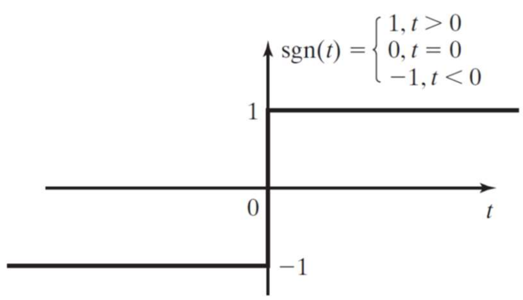
The differentiation property can be stated more generally for the nth derivative as

$$\frac{d^n[f(t)]}{dt^n} \overset{\mathscr{F}}{\longleftrightarrow} (j\omega)^n F(\omega). \tag{5.20}$$

EXAMPLE 5.10 **Fourier transform of the signum function**

We now find the Fourier transform of the signum function shown in Figure 5.11(a):

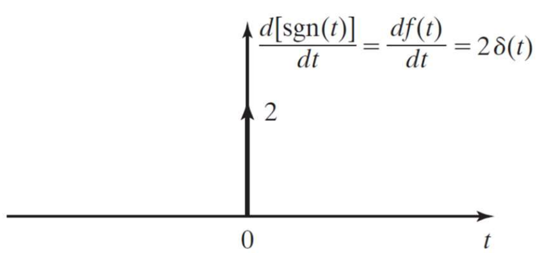
$$f(t) = \text{sgn}(t).$$



(a)

The derivation of $F(\omega)$ is simplified by means of the differentiation property. The time derivative of $\text{sgn}(t)$ is shown in Figure 5.11(b) and is given by

$$\frac{d[f(t)]}{dt} = 2\delta(t).$$



(b)

Because $\delta(t) \overset{\mathscr{F}}{\longleftrightarrow} 1$ (5.8),

$$j\omega F(\omega) = 2.$$

$$\operatorname{sgn}(t) \overset{\mathscr{F}}{\longleftrightarrow} \frac{2}{j\omega}. \qquad (5.21)$$

Time Integration

If

$$f(t) \overset{\mathscr{F}}{\longleftrightarrow} F(\omega),$$

then

$$g(t) = \int_{-\infty}^{t} f(\tau)d\tau \overset{\mathscr{F}}{\longleftrightarrow} \frac{F(\omega)}{j\omega} + \pi F(0)\delta(\omega) = G(\omega), \qquad (5.24)$$

where

$$F(0) = F(\omega)\Big|_{\omega=0} = \int_{-\infty}^{\infty} f(t)dt,$$

EXAMPLE 5.12 **The time-integration property of the Fourier transform**

$$x(t) = A\,\mathrm{rect}\left[\frac{t + t_1/2}{t_1}\right] - A\,\mathrm{rect}\left[\frac{t - t_1/2}{t_1}\right]$$

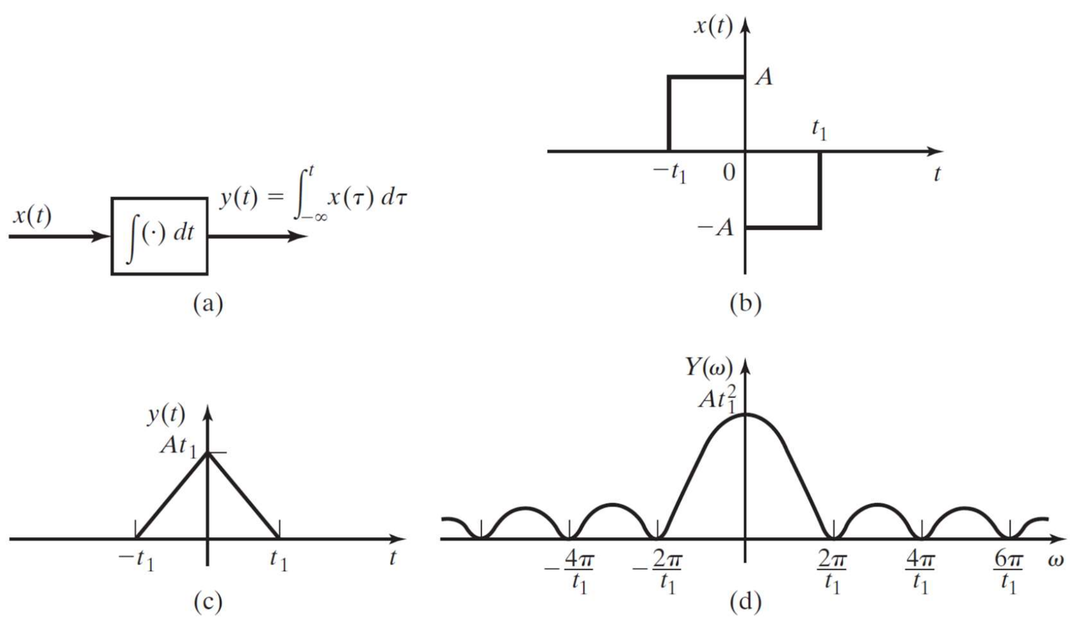


Figure 5.13 System and waveforms for Example 5.12.

$$\begin{aligned} X(\omega) &= At_1 \operatorname{sinc}(t_1\omega/2)[e^{j\omega t_1/2} - e^{-j\omega t_1/2}] \\ &= 2jAt_1 \operatorname{sinc}(t_1\omega/2)\sin(t_1\omega/2) \\ &= j\omega A t_1^2 \operatorname{sinc}(t_1\omega/2)\left[\frac{\sin(t_1\omega/2)}{t_1\omega/2}\right] \\ &= j\omega A t_1^2 \operatorname{sinc}^2(t_1\omega/2). \end{aligned}$$

Next, we use the time-integration property to find

$$Y(\omega) = \frac{1}{j\omega}X(\omega) + \pi X(0)\delta(\omega),$$

where $X(0) = 0$, as can be determined from the previous equation or by finding the time-average value of the signal shown in Figure 5.13(b). Therefore, the frequency spectrum of the output signal is given by

$$Y(\omega) = At_1^2 \operatorname{sinc}^2(t_1\omega/2).$$

Triangular waveforms, such as the one shown in Figure 5.13(c), are sometimes generalized and named as functions. There is no universally accepted nomenclature for these triangular waveforms. We define the triangular pulse as

$$\operatorname{tri}(t/T) = \begin{cases} 1 - \dfrac{|t|}{T}, & |t| < T \\ 0, & |t| \geqq T \end{cases}.$$

$$\operatorname{tri}(t/T) \overset{\mathscr{F}}{\longleftrightarrow} T \operatorname{sinc}^2(T\,\omega/2). \tag{5.27}$$

Frequency Differentiation

The time-differentiation property given by (5.20) has a dual for the case of differentiation in the frequency domain. If

$$f(t) \overset{\mathscr{F}}{\longleftrightarrow} F(\omega),$$

then

$$(-jt)^n f(t) \overset{\mathscr{F}}{\longleftrightarrow} \frac{d^n F(\omega)}{d\omega^n}. \tag{5.28}$$

DC Level

Equation (5.9) gives the transform pair

$$e^{j\omega_0 t} \stackrel{\mathcal{F}}{\longleftrightarrow} 2\pi\delta(\omega - \omega_0).$$

If we allow $\omega_0 = 0$, we have

$$1 \stackrel{\mathcal{F}}{\longleftrightarrow} 2\pi\delta(\omega), \qquad (5.29)$$

which, along with the linearity property, allows us to write the Fourier transform of a dc signal of any magnitude:

$$K \stackrel{\mathcal{F}}{\longleftrightarrow} 2\pi K\delta(\omega). \qquad (5.30)$$

By comparing this transform pair with that of an impulse function in the time domain,

[eq(5.8)] $$\delta(t) \stackrel{\mathcal{F}}{\longleftrightarrow} 1,$$

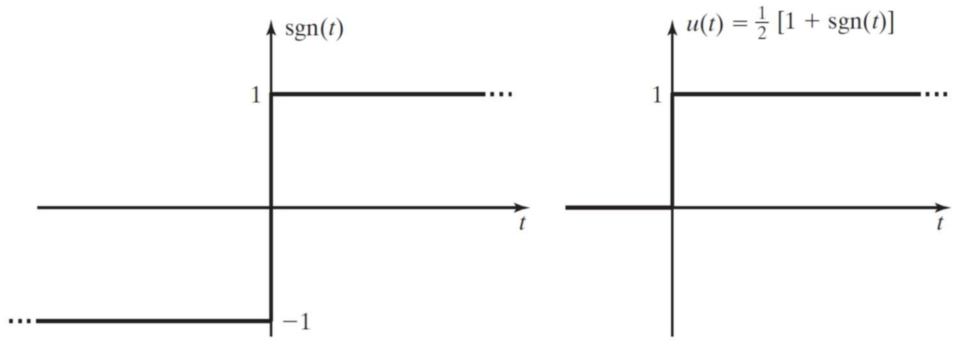
Unit Step Function

The Fourier transform of the unit step function can be derived easily by a consideration of the Fourier transform of the signum function developed in (5.21):

[eq(5.21)] $$\text{sgn}(t) \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{2}{j\omega}.$$

As illustrated in Figure 5.14, the unit step function can be written in terms of the signum function:

$$u(t) = \tfrac{1}{2}[1 + \text{sgn}(t)].$$



$$u(t) \stackrel{\mathscr{F}}{\longleftrightarrow} \pi\delta(\omega) + \frac{1}{j\omega}. \tag{5.31}$$

Exponential Pulse

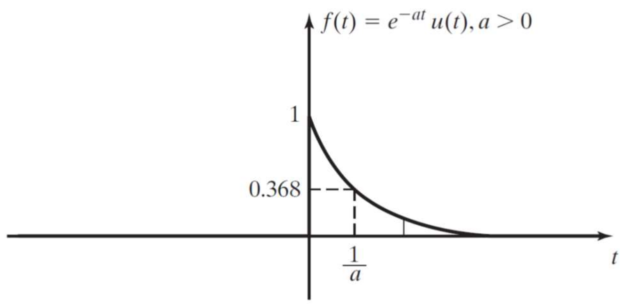
The signal $f(t) = e^{-at}u(t)$, $a > 0$, is shown in Figure 5.17(a). The Fourier transform of this signal will be derived directly from the defining Equation (5.1):

$$F(\omega) = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-j\omega t}dt = \int_{0}^{\infty} e^{-(a+j\omega)t}dt = \frac{1}{a + j\omega}.$$

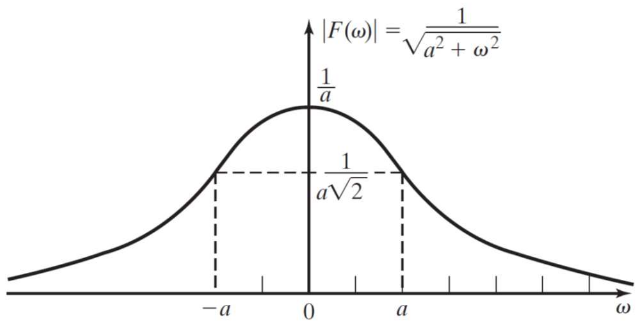
The frequency spectra of this signal are shown in Figures 5.17(b) and (c).

It can be shown that this derivation applies also for a complex, with $\mathrm{Re}\{a\} > 0$. Therefore, the transform pair can be written as

$$e^{-at}u(t), \quad \mathrm{Re}\{a\} > 0 \stackrel{\mathscr{F}}{\longleftrightarrow} \frac{1}{a + j\omega}. \tag{5.34}$$



(a)



(b)

Fourier Transforms of Periodic Functions

[eq(4.11)] $$f(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t},$$

where

[eq(4.23)] $$C_k = \frac{1}{T_0}\int_{T_0} f(t)e^{-jk\omega_0 t}dt.$$

We now will derive a method of determining the Fourier transform of periodic signals.

By (5.1), the Fourier transform of (4.11) yields

$$F(\omega) = \int_{-\infty}^{\infty}\left[\sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t}\right]e^{-j\omega t}dt = \sum_{k=-\infty}^{\infty} C_k \int_{-\infty}^{\infty}(e^{jk\omega_0 t})e^{-j\omega t}dt.$$

From (5.9) and (5.10),

$$\sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t} \overset{\mathscr{F}}{\longleftrightarrow} 2\pi \sum_{k=-\infty}^{\infty} C_k \delta(\omega - k\omega_0). \tag{5.35}$$

Example

$$g(t) = \begin{cases} f(t), & -T_0/2 \leqq t \leqq T_0/2 \\ 0 & \text{elsewhere} \end{cases}, \tag{5.36}$$

$$f(t) = \sum_{n=-\infty}^{\infty} g(t - nT_0). \tag{5.37}$$

Because from (3.18),

$$g(t)*\delta(t - t_0) = g(t - t_0),$$

$$f(t) = \sum_{n=-\infty}^{\infty} g(t)*\delta(t - nT_0) = g(t)*\sum_{n=-\infty}^{\infty}\delta(t - nT_0).$$

The train of impulse functions is expressed by its Fourier series

$$\sum_{n=-\infty}^{\infty}\delta(t - nT_0) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t},$$

where

$$C_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2}\left[\sum_{m=-\infty}^{\infty}\delta(t - mT_0)\right]e^{-jn\omega_0 t}dt.$$

Within the limits of integration, the impulse function will be nonzero only for $m = 0$. Therefore,

$$C_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2}\delta(t)dt = \frac{1}{T_0}.$$

Hence, according to the convolution property of the Fourier transform (5.16),

$$F(\omega) = G(\omega)\mathscr{F}\left\{\frac{1}{T_0}\sum_{n=-\infty}^{\infty}e^{jn\omega_0 t}\right\} = \frac{2\pi}{T_0}G(\omega)\sum_{n=-\infty}^{\infty}\delta(\omega - n\omega_0),$$

5.4 SAMPLING CONTINUOUS-TIME SIGNALS

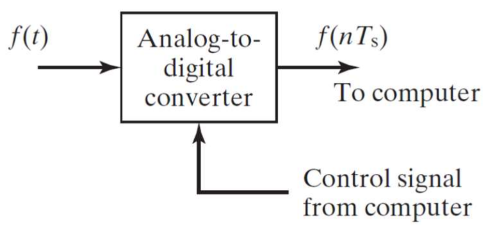


Figure 5.20 An analog-to-digital converter.

Impulse Sampling

The *ideal impulse sampling* operation is modeled by Figure 5.21 and is seen to be a modulation process (modulation is discussed in Chapter 6) in which the carrier signal $\delta_T(t)$ is defined as the train of impulse functions:

$$\delta_T(t) = \sum_{n=-\infty}^{\infty}\delta(t - nT_S). \quad (5.39)$$

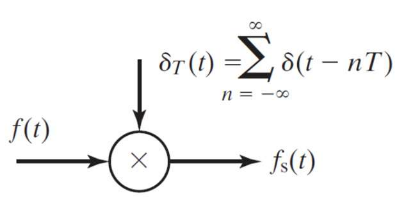


Figure 5.21 Impulse sampling.

$$f_s(t) = f(t)\delta_T(t) = f(t)\sum_{n=-\infty}^{\infty}\delta(t - nT_S) = \sum_{n=-\infty}^{\infty}f(nT_S)\delta(t - nT_S). \quad (5.40)$$

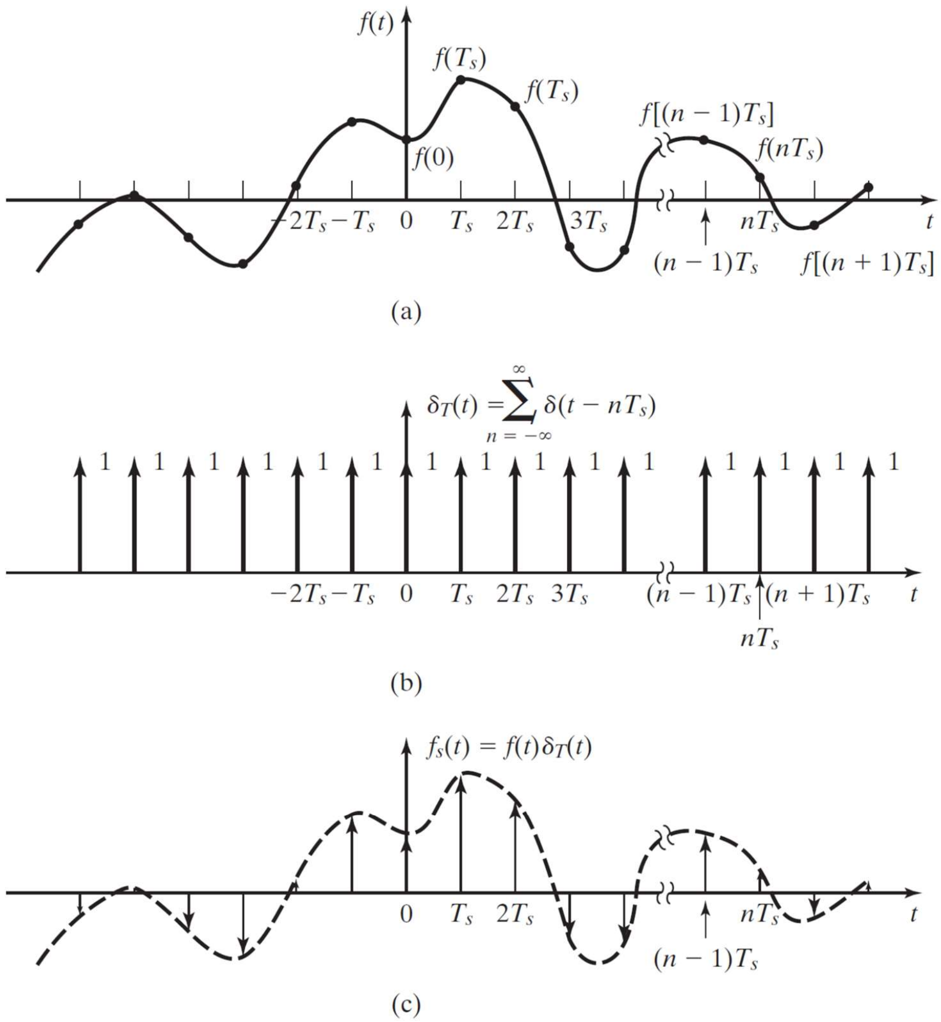


Figure 5.22 Generation of a sampled-data signal.

$$F_s(\omega) = \frac{1}{2\pi}F(\omega)*\left[\omega_s\sum_{k=-\infty}^{\infty}\delta(\omega - k\omega_S)\right] = \frac{1}{T_s}\sum_{k=-\infty}^{\infty}F(\omega)*\delta(\omega - k\omega_S). \quad (5.42)$$

Recall that because of the convolution property of the impulse function [see (3.23)],

$$F(\omega)*\delta(\omega - k\omega_S) = F(\omega - k\omega_S).$$

Thus, the Fourier transform of the impulse-modulated signal (5.40) is given by

$$F_s(\omega) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty}F(\omega - k\omega_S). \quad (5.43)$$

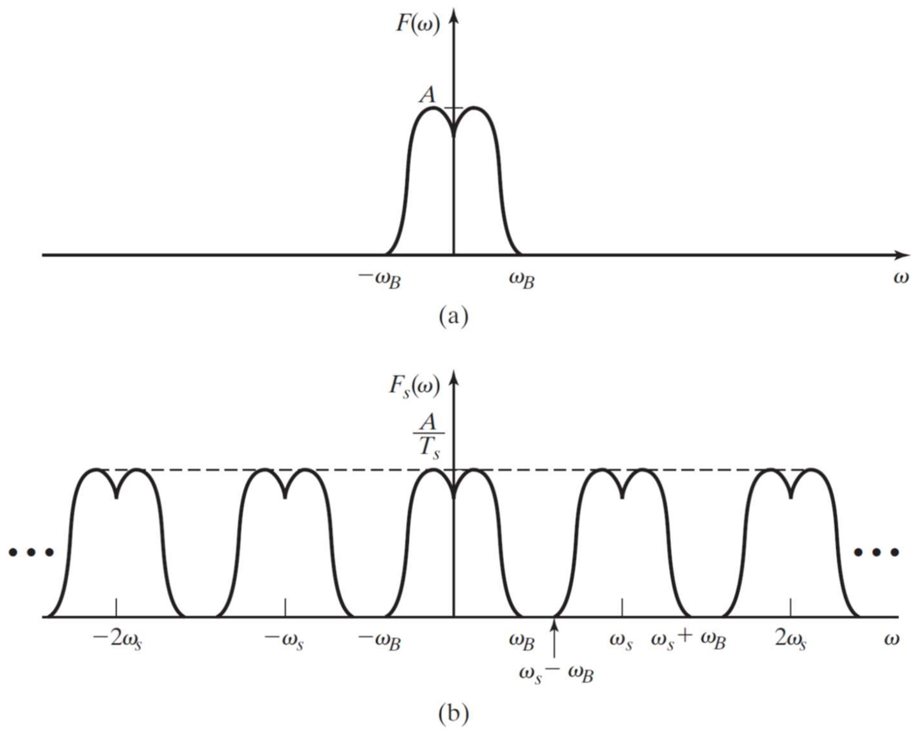


Figure 5.23 The frequency spectrum of a sampled-data signal.

Shannon's Sampling Theorem

A function of time $f(t)$, that contains no frequency components greater than f_M hertz is determined uniquely by the values of $f(t)$ at any set of points spaced

$T_M/2$ ($T_M = 1/f_M$) seconds apart. Hence, according to Shannon's sampling theorem, we must take at least two samples per cycle of the highest frequency component in $f(t)$.

Frequency Response of Linear Systems

Fourier transforms can be used to simplify the calculation of the response of linear systems to input signals. For example, Fourier transforms allow the use of algebraic equations to analyze systems that are described by linear, time-invariant differential equations.

Consider the simple circuit shown in Figure 5.25(a), where $v_1(t)$ is the input signal and $v_2(t)$ is the output signal. This circuit can be described by the differential equations

$$v_1(t) = Ri(t) + L\frac{di(t)}{dt} \quad \text{and} \quad v_2(t) = L\frac{di(t)}{dt}.$$

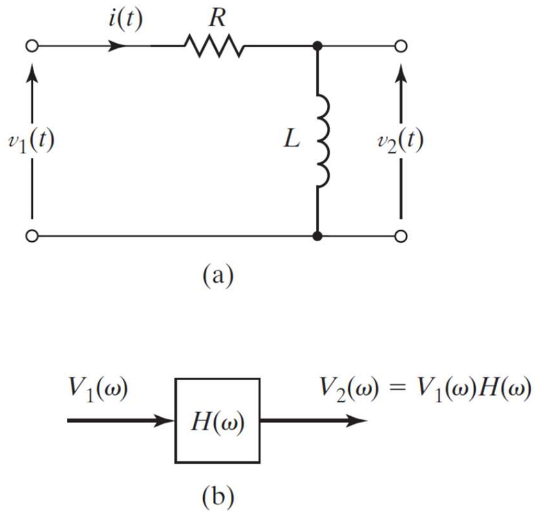


Figure 5.25 An electrical network and its block diagram.

If we take the Fourier transform of each equation, using the properties of linearity and time derivative, we get

$$V_1(\omega) = RI(\omega) + j\omega LI(\omega) \quad \text{and} \quad V_2(\omega) = j\omega LI(\omega).$$

From the first equation, we solve algebraically for $I(\omega)$:

$$I(\omega) = \frac{1}{R + j\omega L}V_1(\omega).$$

Substituting this result into the second equation yields

$$V_2(\omega) = \frac{j\omega L}{R + j\omega L}V_1(\omega),$$

which relates the output voltage of the system to the input voltage.

We define a function

$$H(\omega) = \frac{j\omega L}{R + j\omega L} \tag{5.44}$$

and write the input–output relationship for the system as

$$V_2(\omega) = H(\omega)V_1(\omega), \tag{5.45}$$

or

$$H(\omega) = \frac{V_2(\omega)}{V_1(\omega)}. \tag{5.46}$$

Because the quantity $H(\omega)$ determines the output of the circuit for any given input signal, it is commonly called the *transfer function* of the system. The relationship of (5.45) is illustrated in Figure 5.25(b).

6.1 IDEAL FILTERS

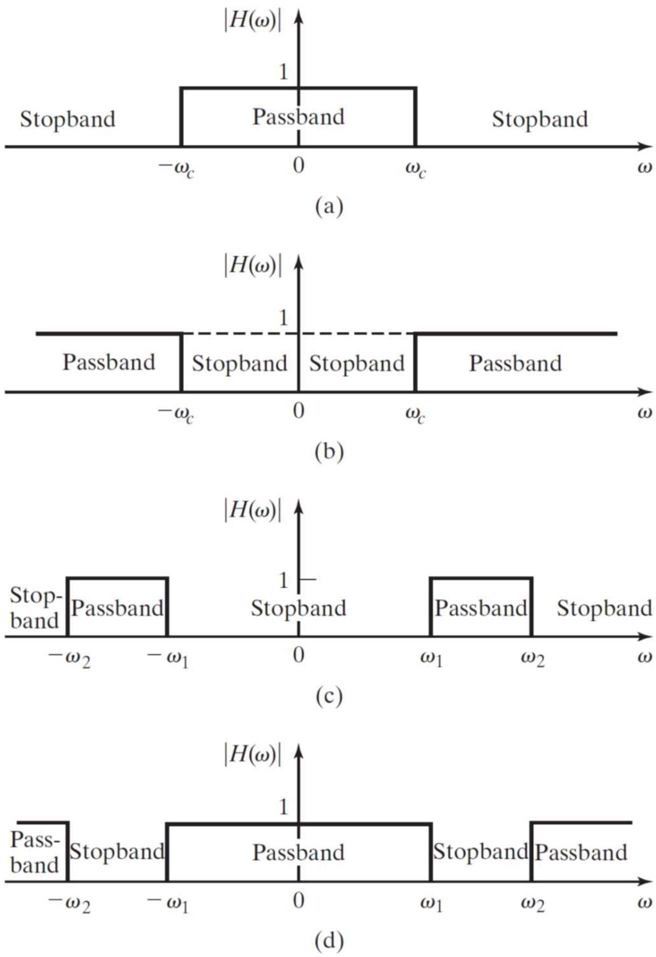


Figure 6.1 Frequency responses of four types of ideal filters.

EXAMPLE 6.1 Application of an ideal high-pass filter

Two signals,

$$g_1(t) = 2\cos(200\pi t) \quad \text{and} \quad g_2(t) = 5\cos(1000\pi t),$$

have been multiplied together as described in Example 5.9. The product is the signal

$$g_3(t) = 5\cos(1200\pi t) + 5\cos(800\pi t).$$

For this example, assume that a certain application requires

$$g_4(t) = 3\cos(1200\pi t).$$

This can be obtained from $g_3(t)$ by a high-pass filter. The Fourier transform of $g_4(t)$ is found, from Table 5.2, to be

$$G_4(\omega) = 3\pi[\delta(\omega - 1200\pi) + \delta(\omega + 1200\pi)].$$

Similarly, the Fourier transform of $g_3(t)$ is found by Table 5.2 and the linearity property:

$$G_3(\omega) = 5\pi[\delta(\omega - 800\pi) + \delta(\omega + 800\pi)] + 5\pi[\delta(\omega - 1200\pi) + \delta(\omega + 1200\pi)].$$

The frequency spectra of $g_4(t)$ and $g_3(t)$ are shown in Figure 6.4(a) and (b), respectively. It can be seen that if the frequency components of $G_3(\omega)$ at $\omega = \pm 1200\pi$ are multiplied by 0.6, and if the frequency components at $\omega = \pm 800\pi$ are multiplied by zero, the result will be the desired signal, $G_4(\omega)$. An ideal high-pass filter that will accomplish this is shown in Figure 6.4(c). The filtering process can be written mathematically as

$$G_4(\omega) = G_3(\omega)H_1(\omega),$$

where

$$H_1(\omega) = 0.6[1 - \text{rect}(\omega/2\omega_c)], \quad 800\pi < \omega_c < 1200\pi.$$ ■

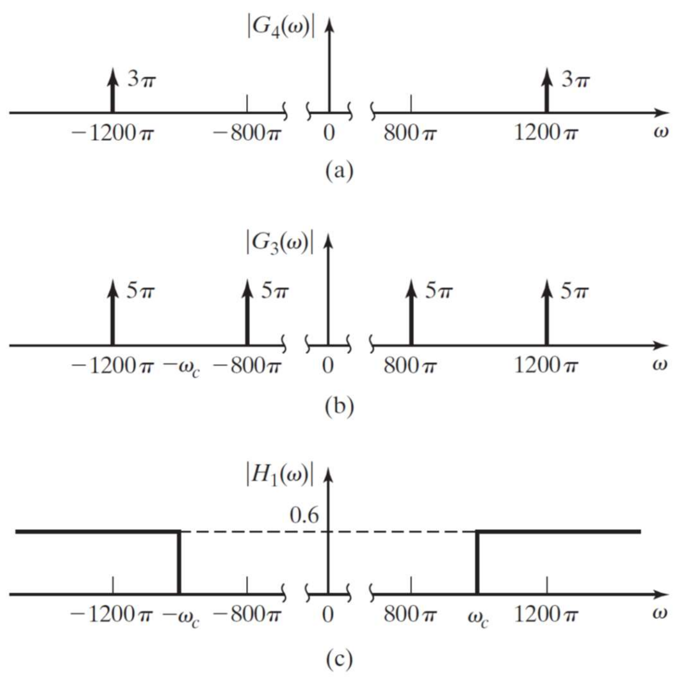


Figure 6.4 Figure for Example 6.1.

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V. Laplace Transform

7.1 DEFINITIONS OF LAPLACE TRANSFORMS

We begin by defining the direct Laplace transform and the inverse Laplace transform. We usually omit the term *direct* and call the direct Laplace transform simply the Laplace transform. By definition, the (*direct*) *Laplace transform* $F(s)$ of a time function is $f(t)$ given by the integral

$$\mathcal{L}_b[f(t)] = F_b(s) = \int_{-\infty}^{\infty} f(t)e^{-st}dt, \tag{7.1}$$

where $\mathcal{L}_b[\cdot]$ indicates the Laplace transform. Definition (7.1) is called the *bilateral*, or *two-sided*, *Laplace transform*—hence, the subscript b. Notice that the bilateral Laplace transform integral becomes the Fourier transform integral if s is replaced by $j\omega$. The Laplace transform variable is complex, $s = \sigma + j\omega$. We can rewrite (7.1) as

$$F(s) = \int_{\infty}^{\infty} f(t)e^{-(\sigma+j\omega)t}dt = \int_{-\infty}^{\infty}(f(t)e^{-\sigma t})e^{-j\omega}dt$$

to show that the bilateral Laplace transform of a signal $f(t)$ can be interpreted as the Fourier transform of that signal multiplied by an exponential function $e^{-\sigma t}$.

The inverse *Laplace taransform* is given by

$$f(t) = \mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} F(s)e^{st}\,ds, j = \sqrt{-1}, \tag{7.2}$$

We now modify Definition (7.1) to obtain a form of the Laplace transform that is useful in many applications. First, we express (7.1) as

$$\mathcal{L}_b[f(t)] = F_b(s) = \int_{-\infty}^{0} f(t)e^{-st}dt + \int_{0}^{\infty} f(t)e^{-st}\,dt. \tag{7.3}$$

Next, we define $f(t)$ to be zero for $t < 0$, such that the first integral in (7.3) is zero. The resulting transform, called the *unilateral*, or *single-sided Laplace transform*, is given by

$$\mathcal{L}[f(t)] = F(s) = \int_{0}^{\infty} f(t)e^{-st}\,dt, \tag{7.4}$$

The Laplace-transform variable s is complex, and we denote its real part as σ and its imaginary part as ω; that is,

$$s = \sigma + j\omega.$$

Figure 7.1 shows the complex plane commonly called the s-plane.

If $f(t)$ is Laplace transformable [if the integral in (7.4) exists], evaluation of (7.4) yields a function $F(s)$. Evaluation of the inverse transform with $F(s)$, using the complex inversion integral, (7.2), then yields $f(t)$. We denote this relationship with

$$f(t) \overset{\mathcal{L}}{\longleftrightarrow} F(s). \tag{7.5}$$

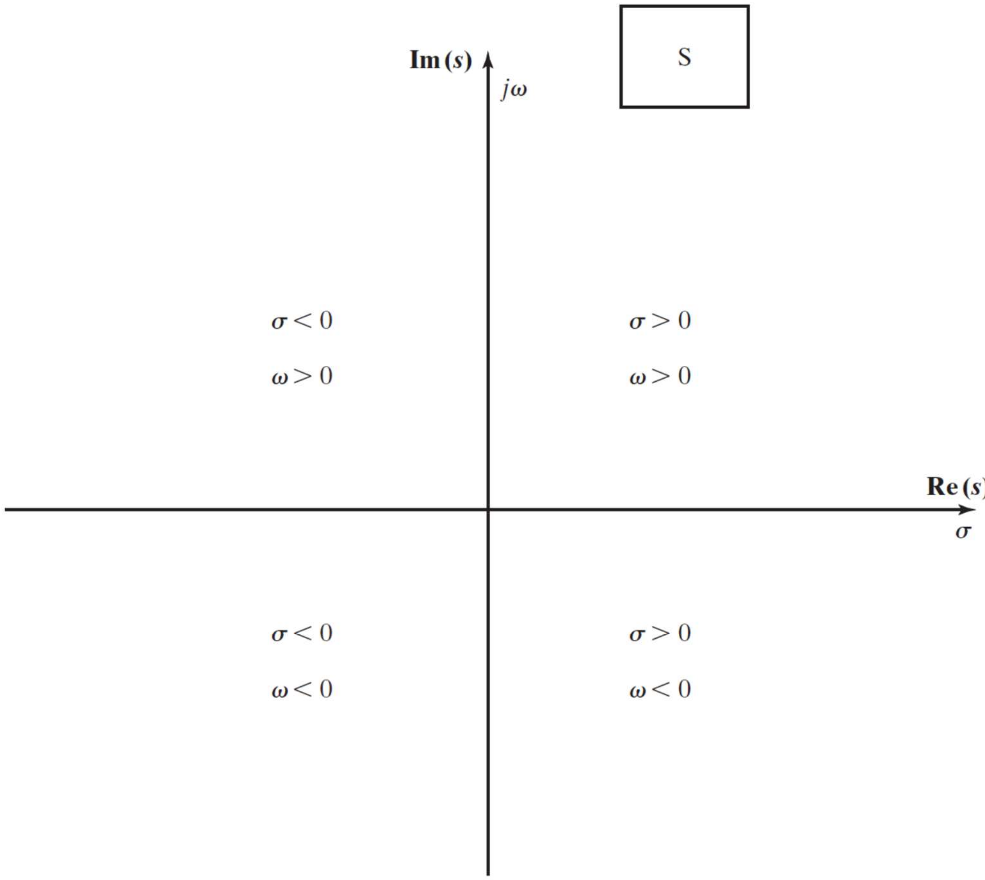


Figure 7.1 The s-plane.

EXAMPLE 7.1 **Laplace transform of a unit step function**

The Laplace transform of the unit step function is now derived for the step occurring at $t = 0$. From (7.4) and (7.12),

$$\mathcal{L}[u(t)] = \int_0^\infty u(t)e^{-st}dt = \int_0^\infty e^{-st}dt$$
$$= \left.\frac{e^{-st}}{-s}\right|_0^\infty = \frac{-1}{s}\left[\lim_{t\to\infty} e^{-st} - 1\right].$$

Hence, the Laplace transform of the unit step function exists *only* if the real part of s is greater than zero. We denote this by

$$\mathcal{L}[u(t)] = \frac{1}{s}, \quad \text{Re}(s) > 0,$$

EXAMPLE 7.2 **Laplace transform of an exponential function**

We next derive the Laplace transform of the exponential function $f(t) = e^{-at}$. From (7.4),

$$F(s) = \int_0^\infty e^{-at}e^{-st}dt = \int_0^\infty e^{-(s+a)t}dt$$
$$= \frac{e^{-(s+a)t}}{-(s+a)}\bigg|_0^\infty = \frac{-1}{s+a}\left[\lim_{t\to\infty} e^{-(s+a)t} - 1\right].$$

This transform exists only if $\text{Re}(s + a)$ is positive. Hence,

$$\mathcal{L}[e^{-at}] = \frac{1}{s+a}, \quad \text{Re}(s+a) > 0,$$

Laplace transform of an impulse function

$$\mathcal{L}[\delta(t - t_0)] = \int_0^\infty \delta(t - t_0)e^{-st}dt = e^{-st}\bigg|_{t=t_0} = e^{-t_0 s}.$$

Hence, we have the Laplace transform pair

$$\delta(t - t_0) \overset{\mathcal{L}}{\longleftrightarrow} e^{-t_0 s}. \tag{7.19}$$

For the unit impulse function occurring at $t = 0$ $(t_0 = 0)$,

$$\delta(t) \overset{\mathcal{L}}{\longleftrightarrow} 1.$$

Laplace transforms of an sinusoidal functions

$$\cos bt = \frac{e^{jbt} + e^{-jbt}}{2}.$$

Hence,

$$\mathcal{L}[\cos bt] = \tfrac{1}{2}[\mathcal{L}[e^{jbt}] + \mathcal{L}[e^{-jbt}]]$$

by the linearity property, (7.10). Then, from (7.14),

$$\mathcal{L}[\cos bt] = \frac{1}{2}\left[\frac{1}{s - jb} + \frac{1}{s + jb}\right] = \frac{s + jb + s - jb}{2(s - jb)(s + jb)} = \frac{s}{s^2 + b^2}.$$

By the same procedure, because $\sin bt = (e^{jbt} - e^{-jbt})/2j$,

$$\mathcal{L}[\sin bt] = \frac{1}{2j}[\mathcal{L}[e^{jbt}] - \mathcal{L}[e^{-jbt}]] = \frac{1}{2j}\left[\frac{1}{s - jb} - \frac{1}{s + jb}\right]$$
$$= \frac{s + jb - s + jb}{2j(s - jb)(s + jb)} = \frac{b}{s^2 + b^2}.$$

$$e^{-at}\cos bt = e^{-at}\left[\frac{e^{jbt} + e^{-jbt}}{2}\right] = \frac{e^{-(a-jb)t} + e^{-(a+jb)t}}{2};$$

thus,

$$\mathcal{L}[e^{-at}\cos bt] = \frac{1}{2}\left[\frac{1}{s + a - jb} + \frac{1}{s + a + jb}\right]$$

$$= \frac{s + a + jb + s + a - jb}{2(s + a - jb)(s + a + jb)} = \frac{s + a}{(s + a)^2 + b^2}.$$

Note the two transform pairs

$$\cos bt \overset{\mathcal{L}}{\longleftrightarrow} \frac{s}{s^2 + b^2}$$

and

$$e^{-at}\cos bt \overset{\mathcal{L}}{\longleftrightarrow} \frac{s + a}{(s + a)^2 + b^2}.$$

$$e^{-at}\sin bt \overset{\mathcal{L}}{\longleftrightarrow} \frac{b}{(s + a)^2 + b^2}.$$

7.4 LAPLACE TRANSFORM PROPERTIES

In Sections 7.1 through 7.3, two properties were derived for the Laplace transform. These properties are

[eq(7.10)] $$\mathcal{L}[a_1 f_1(t) + a_2 f_2(t)] = a_1 F_1(s) + a_2 F_2(s)$$

and

[eq(7.20)] $$\mathcal{L}[e^{-at}f(t)] = F(s)\Big|_{s\leftarrow s+a} = F(s + a).$$

$$\mathcal{L}[f(t - t_0)u(t - t_0)] = e^{-t_0 s}F(s), \tag{7.22}$$

EXAMPLE 7.4 **Laplace transform of a delayed exponential function**

Consider the exponential function shown in Figure 7.6(a), which has the equation

$$f(t) = 5e^{-0.3t},$$

where t is in seconds. This function delayed by 2 s and multiplied by $u(t-2)$ is shown in Figure 7.6(b); the equation for this delayed exponential function is given by

$$f_1(t) = 5e^{-0.3(t-2)}u(t-2).$$

$$\mathscr{L}[f_1(t)] = F_1(s) = e^{-2s}F(s) = \frac{5e^{-2s}}{s+0.3}.$$ ■

Differentiation

We next consider two of the most useful properties of the Laplace transform, which are related to differentiation and integration. The differentiation property was derived in Section 7.2 and is, from (7.15),

$$\mathscr{L}\left[\frac{df(t)}{dt}\right] = sF(s) - f(0^+). \tag{7.24}$$

Property (7.24) is now extended to higher-order derivatives. The Laplace transform of the second derivative of $f(t)$ can be expressed as

$$\mathscr{L}\left[\frac{d^2f(t)}{dt^2}\right] = \mathscr{L}\left[\frac{df'(t)}{dt}\right], \quad f'(t) = \frac{df(t)}{dt}. \tag{7.25}$$

Then, replacing $f(t)$ with $f'(t)$ in (7.24), we can express (7.25) as

$$\mathscr{L}\left[\frac{d^2f(t)}{dt^2}\right] = s\mathscr{L}[f'(t)] - f'(0^+),$$

$$\mathscr{L}\left[\frac{d^nf(t)}{dt^n}\right] = s^nF(s) - s^{n-1}f(0^+) - s^{n-2}f'(0^+) - \cdots - sf^{(n-2)}(0^+) - f^{(n-1)}(0^+), \tag{7.29}$$

EXAMPLE 7.7 **Illustration of the differentiation property**

Consider the Laplace transform of sin bt, from Table 7.2:

$$\mathcal{L}[\sin bt] = \frac{b}{s^2 + b^2}.$$

Now, sin bt can also be expressed as

$$\sin bt = -\frac{1}{b}\frac{d}{dt}(\cos bt).$$

We use this result to find $\mathcal{L}[\sin bt]$. From the differentiation property (7.24) and Table 7.2,

$$\begin{aligned}\mathcal{L}[\sin bt] &= \mathcal{L}\left[-\frac{1}{b}\frac{d}{dt}(\cos bt)\right] \\ &= -\frac{1}{b}\left[s\mathcal{L}[\cos bt] - \cos bt\Big|_{t\to 0^+}\right] \\ &= -\frac{1}{b}\left[s\frac{s}{s^2+b^2} - 1\right] = \frac{b}{s^2+b^2}.\end{aligned}$$

Integration

The property for the integral of a function $f(t)$ is now derived. Let the function $g(t)$ be expressed by

$$g(t) = \int_0^t f(\tau)\, d\tau.$$

$$\mathcal{L}\left[\int_0^t f(\tau) d\tau\right] = \frac{1}{s}F(s). \qquad (7.31)$$

We illustrate this property with an example.

Illustration of the integration property

Consider the following relationship, for $t > 0$:

$$\int_0^t u(\tau) d\tau = \tau\Big|_0^t = t.$$

The Laplace transform of the unit step function is $1/s$, from Table 7.2. Hence, from (7.31),

$$\mathcal{L}[t] = \mathcal{L}\left[\int_0^t u(\tau)\,d\tau\right] = \frac{1}{s}\mathcal{L}[u(t)] = \frac{1}{s}\frac{1}{s} = \frac{1}{s^2},$$

which is the Laplace transform of $f(t) = t$. Note that this procedure can be extended to find the Laplace transform of t^n, for n any positive integer. ■

Transfer Functions

As stated earlier, we prefer to model continuous-time systems with linear differential equations with constant coefficients. The models are then linear and time invariant. (See Section 3.5.) The general equation for the nth-order LTI model is given by

$$\sum_{k=0}^{n} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{n} b_k \frac{d^k x(t)}{dt^k}, \tag{7.47}$$

$$[a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0]Y(s)$$
$$= [b_n s^n + b_{n-1}s^{n-1} + \cdots + b_1 s + b_0]X(s). \tag{7.48}$$

The system transfer function $H(s)$ is defined as the ratio $Y(s)/X(s)$, from (7.48). Therefore, the transfer function for the model of (7.47) is given by

$$H(s) = \frac{Y(s)}{X(s)} = \frac{b_n s^n + b_{n-1}s^{n-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}. \tag{7.49}$$

EXAMPLE 7.12 **LTI system response using Laplace transforms**

Consider again the RL circuit of Figure 7.8, and let $R = 4\Omega$ and $L = 0.5H$. The loop equation for this circuit is given by

$$0.5\frac{di(t)}{dt} + 4i(t) = v(t).$$

The Laplace transform of the loop equation (ignoring initial conditions) is given by

$$(0.5s + 4)I(s) = V(s).$$

We define the circuit input to be the voltage $v(t)$ and the output to be the current $i(t)$; hence, the transfer function is

$$H(s) = \frac{I(s)}{V(s)} = \frac{1}{0.5s + 4}.$$

Note that we could have written the transfer function directly from the loop equation and Equations (7.47) and (7.49).

Now we let $v(t) = 12u(t)$. The transformed current is given by

$$I(s) = H(s)V(s) = \frac{1}{0.5s + 4}\frac{12}{s} = \frac{24}{s(s + 8)}.$$

The partial-fraction expansion of $I(s)$ is then

$$I(s) = \frac{24}{s(s + 8)} = \frac{k_1}{s} + \frac{k_2}{s + 8},$$

where (see Appendix F)

$$k_1 = s\left[\frac{24}{s(s + 8)}\right]_{s=0} = \left.\frac{24}{s + 8}\right|_{s=0} = 3$$

and

$$k_2 = (s + 8)\left[\frac{24}{s(s + 8)}\right]_{s=-8} = \left.\frac{24}{s}\right|_{s=-8} = -3.$$

Thus,

$$I(s) = \frac{24}{s(s + 8)} = \frac{3}{s} + \frac{-3}{s + 8},$$

and the inverse transform, from Table 7.2, yields

$$i(t) = 3[1 - e^{-8t}]$$

for $t > 0$.

If the numerator and denominator polynomials in (7.49) are presented in product-of-sums form, the transfer function is shown as

$$H(s) = \frac{K(s - z_1)(s - z_2)\cdots(s - z_m)}{(s - p_1)(s - p_2)\cdots(s - p_n)}. \tag{7.50}$$

EXAMPLE 7.13 **Poles and zeros of a transfer function**

A transfer function is given in the form of (7.49) as

$$H(s) = \frac{4s + 8}{2s^2 + 8s + 6}.$$

The transfer function is rewritten in the form of (7.50) as

$$H(s) = \frac{2(s + 2)}{(s + 1)(s + 3)}.$$

We now see that this transfer function has one zero at $s = -2$ and two poles located at $s = -1$ and $s = -3$. The poles and the zero of the transfer function are plotted in the s-plane in Figure 7.10. It is standard practice to plot zeros with the symbol ⦿ and poles with the symbol ×.

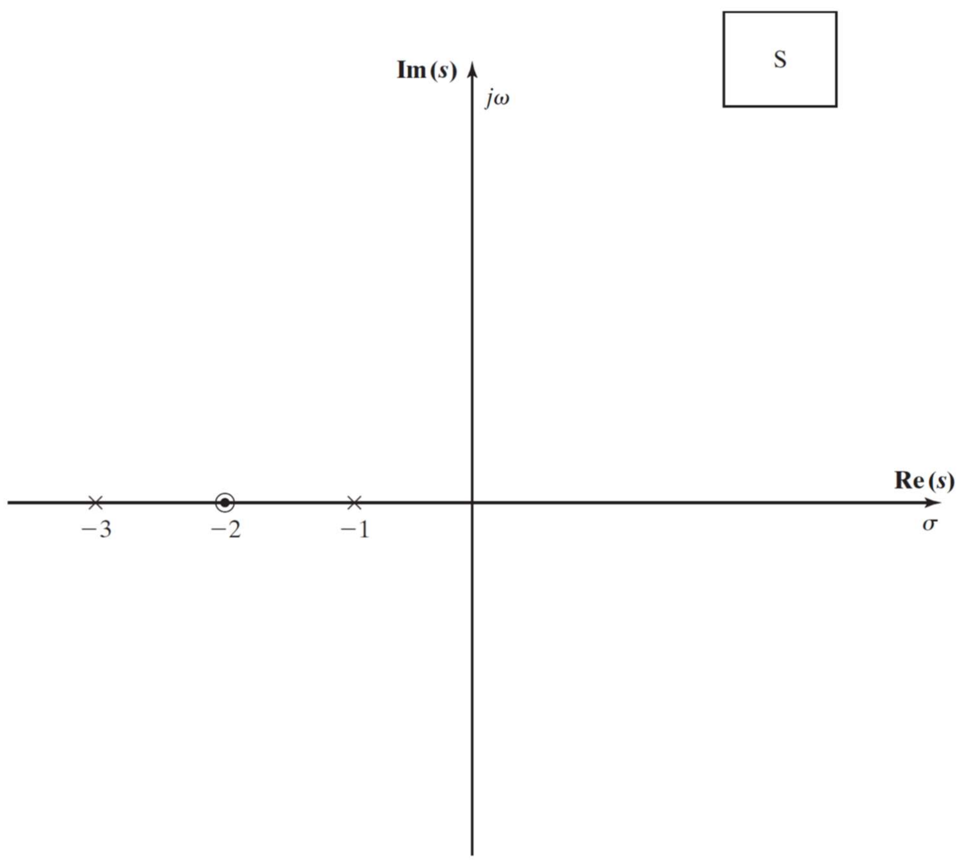


Figure 7.10 ■

Convolution

$$y(t) = x(t)*h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau, \tag{7.51}$$

$$y(t) = h(t)*x(t) \Rightarrow Y(s) = H(s)X(s). \tag{7.55}$$

$$H(s) = \int_{0}^{\infty} h(t)e^{-st}\,dt. \tag{7.56}$$

EXAMPLE 7.14 **Response of LTI system from the impulse response**

The unit step response is calculated for an LTI system with the impulse response $h(t)$ given in Figure 7.12(a). We express this function as

$$h(t) = u(t) - u(t-1).$$

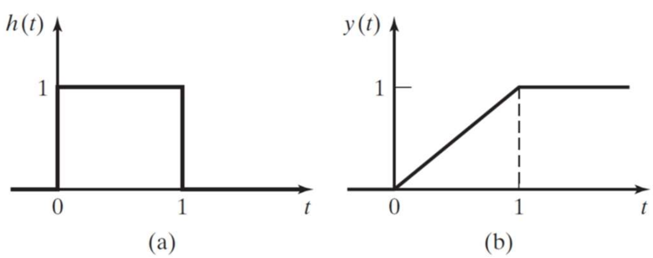


Figure 7.12 Signals for Example 7.14.

Using the real-shifting property, we find the Laplace transform of $h(t)$ to be

$$H(s) = \frac{1 - e^{-s}}{s}.$$

Note that this transfer function is not a rational function. From (7.55), the system output $Y(s)$ is then

$$Y(s) = H(s)X(s) = \frac{1 - e^{-s}}{s}\frac{1}{s} = \frac{1}{s^2}[1 - e^{-s}].$$

From the Laplace transform table and the real-shifting property, we find the system output to be

$$y(t) = tu(t) - [t-1]u(t-1).$$

EXAMPLE 7.16 **Inverse transform involving repeated poles**

The unit step response of a system with the third-order transfer function

$$\frac{Y(s)}{X(s)} = H(s) = \frac{4s^2 + 4s + 4}{s^3 + 3s^2 + 2s}$$

will be found. Hence, $x(t) = u(t)$ and $X(s) = 1/s$. The system output is then

$$Y(s) = H(s)X(s) = \frac{4s^2 + 4s + 4}{s^3 + 3s^2 + 2s}\left(\frac{1}{s}\right).$$

Because this function is not in Table 7.2, we must find its partial-fraction expansion:

$$Y(s) = \frac{4s^2 + 4s + 4}{s^2(s + 1)(s + 2)} = \frac{k_1}{s^2} + \frac{k_2}{s} + \frac{k_3}{s + 1} + \frac{k_4}{s + 2}.$$

We solve first for k_1, k_3, and k_4:

$$k_1 = \left.\frac{4s^2 + 4s + 4}{(s + 1)(s + 2)}\right|_{s=0} = \frac{4}{2} = 2;$$

$$k_3 = \left.\frac{4s^2 + 4s + 4}{s^2(s + 2)}\right|_{s=-1} = \frac{4 - 4 + 4}{(1)(1)} = 4;$$

$$k_4 = \left.\frac{4s^2 + 4s + 4}{s^2(s + 1)}\right|_{s=-1} = \frac{16 - 8 + 4}{(4)(-1)} = -3.$$

We calculate k_2 by Equation (F.8) of Appendix F:

$$k_2 = \frac{d}{ds}[s^2Y(s)]_{s=0} = \frac{d}{ds}\left[\frac{4s^2 + 4s + 4}{s^2 + 3s + 2}\right]_{s=0}$$

$$= \left.\frac{(s^2 + 3s + 2)(8s + 4) - (4s^2 + 4s + 4)(2s + 3)}{[s^2 + 3s + 2]^2}\right|_{s=0}$$

$$= \frac{(2)(4) - (4)(3)}{4} = -1.$$

The partial-fraction expansion is then

$$Y(s) = \frac{4s^2 + 4s + 4}{s^2(s^2 + 3s + 2)} = \frac{2}{s^2} + \frac{-1}{s} + \frac{4}{s + 1} + \frac{-3}{s + 2},$$

which yields the output signal

$$y(t) = 2t - 1 + 4e^{-t} - 3e^{-2t},$$

7.7 LTI SYSTEMS CHARACTERISTICS

In this section, we consider the properties of causality, stability, invertibility, and frequency response for LTI systems, relative to the Laplace transform.

Causality

The unilateral Laplace transform requires that any time function be zero for $t < 0$. Hence, the impulse response $h(t)$ must be zero for negative time. Because this is also the requirement for causality, the unilateral transform can be applied to causal systems only. The bilateral Laplace transform, introduced in Section 7.8, must be employed for noncausal systems.

Stability

We now relate bounded-input bounded-output (BIBO) stability to transfer functions. Recall the definition of BIBO stability:

> **BIBO Stability**
> A system is stable if the output remains bounded for all time for any bounded input.

We express the transfer function of an nth-order system as

[eq(7.49)] $$H(s) = \frac{Y(s)}{X(s)} = \frac{b_n s^n + b_{n-1}s^{n-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0},$$

where $a_n \neq 0$. The denominator of this transfer function can be factored as

$$a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0 = a_n(s - p_1)(s - p_2)\cdots(s - p_n). \quad (7.60)$$

The zeros of this polynomial are the *poles* of the transfer function, where, by definition, the poles of a function $H(s)$ are the values of s at which $H(s)$ is unbounded.

We can express the output $Y(s)$ in (7.49) as

$$\begin{aligned} Y(s) &= \frac{b_n s^n + b_{n-1}s^{n-1} + \cdots + b_1 s + b_0}{a_n(s - p_1)(s - p_2)\cdots(s - p_n)} X(s), \\ &= \frac{k_1}{s - p_1} + \frac{k_2}{s - p_2} + \cdots + \frac{k_n}{s - p_n} + Y_x(s), \end{aligned} \quad (7.61)$$

The inverse transform of (7.61) yields

$$\begin{aligned} y(t) &= k_1 e^{p_1 t} + k_2 e^{p_2 t} + \cdots + k_n e^{p_n t} + y_x(t) \\ &= y_c(t) + y_x(t). \end{aligned} \quad (7.62)$$

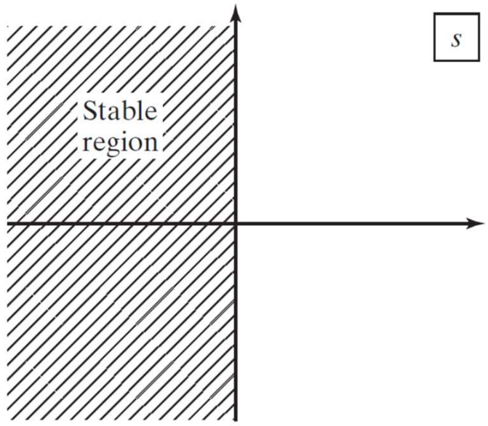


Figure 7.13 Stable region for poles of $H(s)$.

We see from the preceding discussion that an LTI system is stable, provided that all poles of the system transfer function are in the left half of the s-plane—that is, provided that $\text{Re}(p_i) < 0, i = 1, 2, \ldots, n$. Recall that we derived this result in Section 3.6 by taking a different approach. The stable region for the poles of $H(s)$ in the s-plane is illustrated in Figure 7.13.

EXAMPLE 7.17 **Stability of an LTI system**

A much-simplified transfer function for the booster stage of the Saturn V rocket, used in trips to the moon, is given by

$$H(s) = \frac{0.9402}{s^2 - 0.0297} = \frac{0.9402}{(s + 0.172)(s - 0.172)},$$

where the system input was the engine thrust and the system output was the angle of the rocket relative to the vertical. The system modes are $e^{-0.172t}$ and $e^{0.172t}$; the latter mode is obviously unstable. A control system was added to the rocket, such that the overall system was stable and responded in an acceptable manner. ■

Invertibility

We restate the definition of the inverse of a system from Section 2.7 in terms of transfer functions.

> **Inverse of a System**
> The inverse of an LTI system $H(s)$ is a second system $H_i(s)$ that, when cascaded with $H(s)$, yields the identity system.

Thus, $H_i(s)$ is defined by the equation

$$H(s)H_i(s) = 1 \Rightarrow H_i(s) = \frac{1}{H(s)}. \quad (7.64)$$

$$H(s) = \frac{b_n s^n + b_{n-1}s^{n-1} + \cdots + b_0}{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_0}. \quad (7.65)$$

Hence, the inverse system has the transfer function

$$H_i(s) = \frac{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_0}{b_n s^n + b_{n-1}s^{n-1} + \cdots + b_0}. \quad (7.66)$$

Frequency Response

Recall from (5.1) the definition of the Fourier transform:

$$F(\omega) = \mathscr{F}[f(t)] = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\, dt. \quad (7.67)$$

From Section 5.5, using the Fourier transform, we find that the transfer function for a causal system with the impulse response $h(t)$ is given by

$$H_f(\omega) = \mathscr{F}[h(t)] = \int_{0}^{\infty} h(t)e^{-j\omega t}\, dt. \quad (7.68)$$

Comparing this transfer function with that based on the Laplace transform, namely,

$$H_l(s) = \mathscr{L}[h(t)] = \int_{0}^{\infty} h(t)e^{-st}\, dt, \quad (7.69)$$

we see that the two transfer functions are related by

$$H_f(\omega) = H_l(s)\Big|_{s=j\omega} = H_l(j\omega). \quad (7.70)$$

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{b_n(j\omega)^n + b_{n-1}(j\omega)^{n-1} + \cdots + b_1(j\omega) + b_0}{a_n(j\omega)^n + a_{n-1}(j\omega)^{n-1} + \cdots + a_1(j\omega) + a_0}. \quad (7.71)$$

VI. Discrete Time Systems

In summary, a discrete-time signal is an ordered sequence of numbers. The sequence is usually expressed as $\{f[n]\}$, where this notation denotes the sequence $\ldots, f[-2], f[-1], f[0], f[1], f[2], \ldots$. We usually consider $f[n]$, for n a noninteger, to be undefined.

Unit Step and Unit Impulse Functions

We begin the study of discrete-time signals by defining two signals. First, the *discrete-time unit step function* $u[n]$ is defined by

$$u[n] = \begin{cases} 1, & n \geqq 0 \\ 0, & n < 0. \end{cases} \tag{9.9}$$

$$u[n - n_0] = \begin{cases} 1, & n \geqq n_0 \\ 0, & n < n_0. \end{cases} \tag{9.10}$$

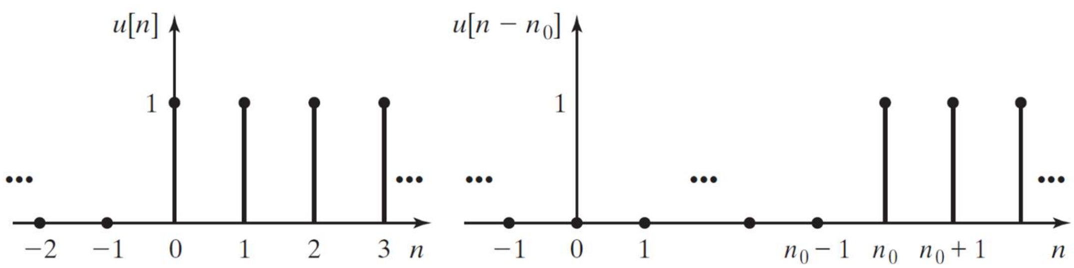


Figure 9.3 Discrete-time unit step functions.

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0. \end{cases} \tag{9.11}$$

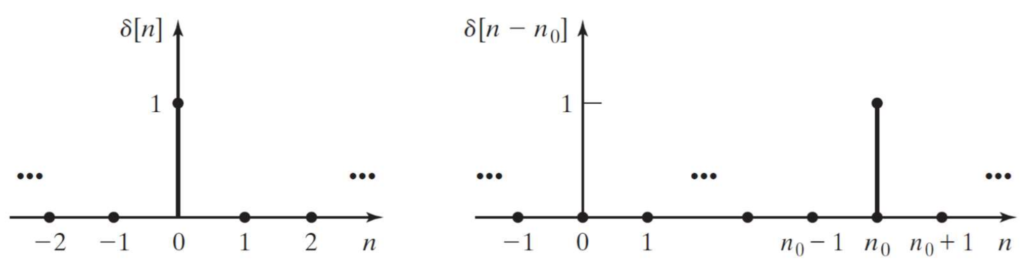


Figure 9.4 Discrete-time unit impulse functions.

$$\delta[n - n_0] = \begin{cases} 1, & n = n_0 \\ 0, & n \neq n_0 \end{cases} \quad (9.13)$$

$$\delta[n] = u[n] - u[n - 1]. \quad (9.12)$$

9.3 CHARACTERISTICS OF DISCRETE-TIME SIGNALS

In Section 2.2, some useful characteristics of continuous-time signals were defined. We now consider the same characteristics for discrete-time signals.

Even and Odd Signals

In this section, we define even and odd signals (functions). A discrete-time signal $x_e[n]$ is *even* if

$$x_e[n] = x_e[-n], \quad (9.25)$$

and the signal $x_o[n]$ is *odd* if

$$x_o[n] = -x_o[-n]. \quad (9.26)$$

Any discrete-time signal $x[n]$ can be expressed as the sum of an even signal and an odd signal:

$$x[n] = x_e[n] + x_o[n]. \quad (9.27)$$

To show this, we replace n with $-n$ to yield

$$x[-n] = x_e[-n] + x_o[-n] = x_e[n] - x_o[n]. \quad (9.28)$$

The sum of (9.27) and (9.28) yields the even part of $x[n]$:

$$x_e[n] = \tfrac{1}{2}(x[n] + x[-n]). \quad (9.29)$$

The subtraction of (9.28) from (9.27) yields the odd part of $x[n]$:

$$x_o[n] = \tfrac{1}{2}(x[n] - x[-n]). \quad (9.30)$$

The *average* value, or *mean* value, of a discrete-time signal is given by

$$A_x = \lim_{N\to\infty} \frac{1}{2N + 1} \sum_{k=-N}^{N} x[k]. \quad (9.31)$$

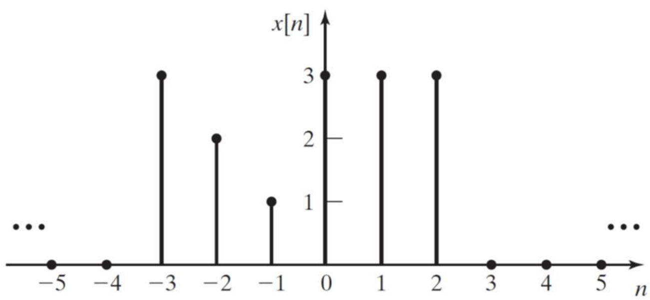
As is the case of continuous-time signals, the average value of a discrete-time signal is contained in its even part, and the average value of an odd signal is always zero.

Even and odd signals have the following properties:

1. The sum of two even signals is even.
2. The sum of two odd signals is odd.
3. The sum of an even signal and an odd signal is neither even nor odd.
4. The product of two even signals is even.
5. The product of two odd signals is even.
6. The product of an even signal and an odd signal is odd.

EXAMPLE 9.5 **Even and odd functions**

The even and the odd parts of the discrete-time signal $x[n]$ of Figure 9.13(a) will be found. Since the signal has only six nonzero values, a strictly mathematical approach is used. Table 9.5 gives the solution, using (9.29) and (9.30). All values not given in this table are zero. The even and odd parts of $x[n]$ are plotted in Figure 9.13.



(a)

TABLE 9.5 Values for Example 9.5

n	$x[n]$	$x[-n]$	$x_e[n]$	$x_o[n]$
−3	3	0	1.5	1.5
−2	2	3	2.5	−0.5
−1	1	3	2	−1
0	3	3	3	0
1	3	1	2	1
2	3	2	2.5	0.5
3	0	3	1.5	−1.5

■

The sum of all values of $x_o[n]$ is zero, since, for any value of n, from (9.26) it follows that

$$x_o[n] + x_o[-n] = x_o[n] - x_o[n] = 0.$$

Note also that $x_o[0]$ is always zero.

Signals Periodic in *n*

We now consider periodic discrete-time signals. By definition, a discrete-time signal $x[n]$ is *periodic* with period N if

$$x[n + N] = x[n]. \tag{9.32}$$

Of course, both n and N are integers.

We next consider the discrete-time complex exponential signal that is not necessarily obtained by sampling a continuous-time signal. We express the signal as

$$x[n] = e^{j\Omega_0 n} = 1\underline{/\Omega_0 n}. \tag{9.37}$$

This signal can be represented in the complex plane as a vector of unity magnitude at the angles $\Omega_0 n$, as shown in Figure 9.15. The projection of this vector onto the real axis is $\cos(\Omega_0 n)$ and onto the imaginary axis is $\sin(\Omega_0 n)$, since

$$e^{j\Omega_0 n} = \cos(\Omega_0 n) + j\sin(\Omega_0 n).$$

We now consider (9.37) in a different manner. The complex exponential signal of (9.37) is periodic, provided that

$$x[n] = e^{j\Omega_0 n} = x[n + N] = e^{j(\Omega_0 n + \Omega_0 N)} = e^{j(\Omega_0 n + 2\pi k)}, \tag{9.38}$$

where k is an integer. Thus, periodicity requires that

$$\Omega_0 N = 2\pi k \Rightarrow \Omega_0 = \frac{k}{N}2\pi, \tag{9.39}$$

so that Ω_0 must be expressible as 2π multiplied by a rational number. For example, $x[n] = \cos(2n)$ is *not* periodic, since $\Omega_0 = 2$. The signal $x[n] = \cos(0.1\pi n)$ is periodic, since $\Omega_0 = 0.1\pi$. For this case, $k = 1$, and $N = 20$ satisfies (9.39).

As a final point, from (9.39), the complex exponential signal $e^{j\Omega_0 n}$ is periodic with N samples per period, provided that the integer N satisfies the equation

$$N = \frac{2\pi k}{\Omega_0}. \tag{9.40}$$

In this equation, k is the smallest positive integer that satisfies this equation, such that N is an integer greater than unity. For example, for the signal $x[n] = \cos(0.1\pi n)$, the number of samples per period is

$$N = \frac{2\pi k}{0.1\pi} = 20k = 20, \quad k = 1. \tag{9.41}$$

For a final example, consider the signal $x[n] = \cos(2\pi n)$. Then

$$N = \frac{2\pi k}{2\pi} = k, \quad (9.42)$$

and this equation is satisfied for $N = k = 1$. This signal can be expressed as

$$x[n] = \cos(2\pi n) = 1.$$

Hence, the discrete-time signal is constant.

Signals Periodic in Ω

The conditions for the complex exponential signal $e^{j\Omega_0 n}$ to be periodic in n were just developed. However, this signal is *always* periodic in the discrete-frequency variable Ω. Consider this signal with Ω_0 replaced with $(\Omega_0 + 2\pi)$—that is,

$$e^{j(\Omega_0+2\pi)n} = e^{j\Omega_0 n}e^{j2\pi n} = e^{j\Omega_0 n} \quad (9.43)$$

(since $e^{j2\pi n} = 1$). Hence, the signal $e^{j\Omega_0 n}$ is periodic in Ω with period 2π, independent of the value of Ω_0. Of course, the sinusoidal signal $\cos(\Omega_0 n + \theta)$ is also periodic in Ω with period 2π. This property has a great impact on the sampling of signals, as shown in Chapters 5 and 6.

Note that periodic continuous-time signals are not periodic in frequency. For example, for the complex exponential signal,

$$e^{j(\omega+a)t} = e^{j\omega t}\, e^{jat} \neq e^{j\omega t}, \quad a \neq 0.$$

Discrete-time Systems

We now use an example of a system to introduce a common discrete-time signal. The block shown in Figure 9.17(a) represents a memory device that stores a number. Examples of this device are shift registers or memory locations in a digital computer. Every T seconds, we shift out the number stored in the device. Then a different number is shifted into the device and stored. If we denote the number shifted into the device as $x[n]$, the number just shifted out must be $x[n - 1]$. A device used in this manner is called an *ideal time delay*. The term *ideal* indicates that the numbers are not altered in any way, but are only delayed.

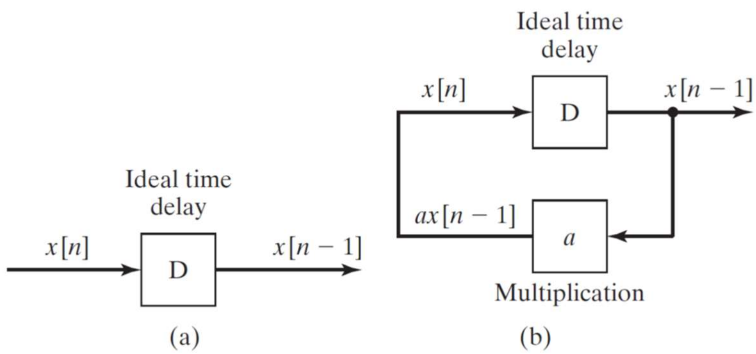


Figure 9.17 Discrete-time system.

$$x[n] = ax[n-1]. \qquad (9.44)$$

Suppose that at the first instant (denoted as $n = 0$), the number unity is stored in the delay; that is, $x[0] = 1$. We now iteratively solve for $x[n]$, $n > 0$, using (9.44) (recall that the ideal time delay outputs its number every T seconds):

$$\begin{aligned} x[1] &= ax[0] = a; \\ x[2] &= ax[1] = a^2; \\ x[3] &= ax[2] = a^3; \\ &\vdots; \\ x[n] &= ax[n-1] = a^n. \end{aligned}$$

Thus, this system generates the signal $x[n] = a^n$ for the initial condition $x[0] = 1$.

$$y[n] = T(x[n]). \qquad (9.59)$$

This notation represents a transformation and not a function; that is, $T(x[n])$ is not a mathematical function into which we substitute $x[n]$ and directly calculate $y[n]$. The set of equations relating the input $x[n]$ and the output $y[n]$ is called a *mathematical model,* or, simply, a *model,* of the system. Given the input $x[n]$, this set of equations must be solved to obtain $y[n]$. For discrete-time systems, the model is usually a set of difference equations.

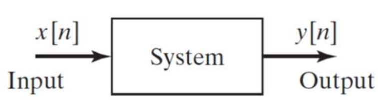


Figure 9.22 Block diagram for a discrete-time system.

Interconnecting Systems

We define two basic connections for systems. The first, the *parallel* connection, is illustrated in Figure 9.23. The circle in this figure denotes the summation of signals.

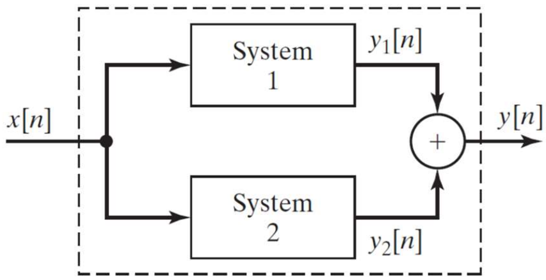


Figure 9.23 Parallel connection of systems.

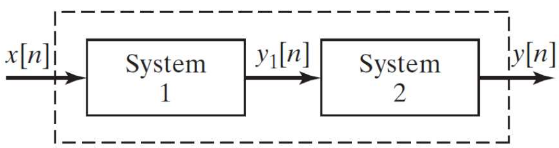


Figure 9.24 Series, or cascade, connection of systems.

Let the output of System 1 be $y_1[n]$ and that of System 2 be $y_2[n]$. The output signal of the total system, $y[n]$, is given by

$$y[n] = y_1[n] + y_2[n] = T_1(x[n]) + T_2(x[n]) = T(x[n]), \qquad (9.61)$$

where $y[n] = T(x[n])$ is the notation for the total system.

The second basic connection for systems is illustrated in Figure 9.24. This connection is called the *series,* or *cascade,* connection. In this figure, the output signal of the first system is $y_1[n] = T_1(x[n])$, and the total system output signal is

$$y[n] = T_2(y_1[n]) = T_2(T_1(x[n])) = T(x[n]). \qquad (9.62)$$

EXAMPLE 9.9 **Interconnection of a discrete system**

Consider the system of Figure 9.25. Each block represents a system, with a number given to identify each system. We can write the following equations for the system:

$$y_3[n] = T_1(x[n]) + T_2(x[n])$$

and

$$y_4[n] = T_3(y_3[n]) = T_3(T_1(x[n]) + T_2(x[n])).$$

Thus,

$$\begin{aligned} y[n] &= y_2[n] + y_4[n] \\ &= T_2(x[n]) + T_3(T_1(x[n]) + T_2(x[n])) = T(x[n]). \end{aligned}$$

This equation denotes only the interconnection of the systems. The mathematical model of the total system will depend on the models of the individual subsystems.

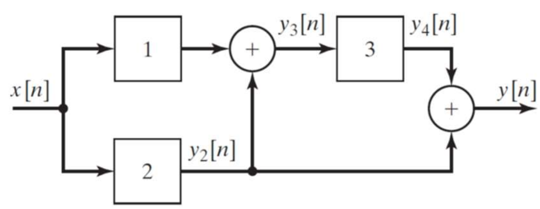


Figure 9.25 System for Example 9.9. ■

9.6 PROPERTIES OF DISCRETE-TIME SYSTEMS

In Section 9.5, the Euler integrator and the α-filter were given as examples of discrete-time systems. In this section, we present some of the characteristics and properties of discrete-time systems.

In the following, $x[n]$ denotes the input of a system and $y[n]$ denotes the output. We show this relationship symbolically by the notation

$$x[n] \rightarrow y[n]. \tag{9.63}$$

As with continuous-time systems, we read this relation as $x[n]$ produces $y[n]$. Relationship (9.63) has the same meaning as

[eq(9.59)] $$y[n] = T(x[n]).$$

The definitions to be given are similar to those listed in Section 2.7 for continuous-time systems.

Systems with Memory

We first define a system that has memory:

Memory
A system has memory if its output at time n_0, $y[n_0]$, depends on input values other than $x[n_0]$. Otherwise, the system is memoryless.

For a discrete signal $x[n]$, time is represented by the discrete increment variable n. An example of a simple memoryless discrete-time system is the equation

$$y[n] = 5x[n].$$

A memoryless system is also called a *static system*.

A system with memory is also called a *dynamic system*. An example of a system with memory is the Euler integrator of (9.5):

$$y[n] = y[n-1] + Hx[n-1].$$

Recall from Section 9.1 and (9.8) that this equation can also be expressed as

$$y[n] = H \sum_{k=-\infty}^{n-1} x[k], \tag{9.64}$$

and we see that the output depends on all past values of the input.

A second example of a discrete system with memory is one whose output is the average of the last two values of the input. The difference equation describing this system is

$$y[n] = \tfrac{1}{2}[x[n] + x[n-1]]. \tag{9.65}$$

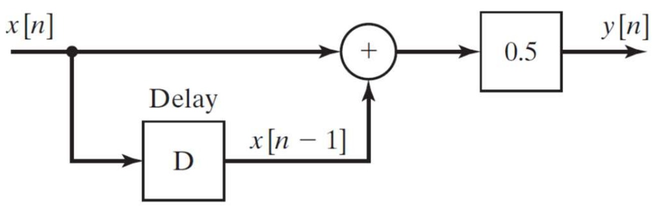


Figure 9.26 Averaging system.

Invertibility

We now define *invertibility*:

> **Invertibility**
> A system is said to be invertible if distinct inputs result in distinct outputs.

A second definition of invertibility is that the input of an invertible system can be determined from its output. For example, the memoryless system described by

$$y[n] = |x[n]|$$

is not invertible. The inputs of +2 and −2 produce the same output of +2.

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VII. Z TRANSFORM

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